**UNIVERSIDAD DEL CEMA**

# *ARBITRAGE PORTFOLIOS*

- **SUFFICIENT CONDITIONS WITHIN CML AND SML CONTEXTS**
- **SUFFICIENT CONDTIONS BY MEANS OF SEPARATION PORTFOLIOS AND TREYNOR LINES**

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## Abstract

It should be expected from this paper an expansion on some distinctive issues regarding arbitrage portfolios: i) a definition on arbitrage portfolios that enables adjustments to SML and CML environments; ii) sufficient conditions to set up arbitrage portfolios against the SML and CML; iii) feasibility of separation portfolios to carry out arbitrage not only against SML but CML as well; iv) arbitrage of portfolios located in Treynor's lines by using separation portfolios within a SML environment.

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## INTRODUCTION

Models not only provide understanding of real world complexities, but also allow taking advantage whenever the values of the underlying working variables depart from the expected values assessed by those same models.

A good example of the functionality of models is given by portfolio theory. Since its inception as a subject on its own in the mainstream of Financial Economics with Markowitz (1952;1959), a string of proposals have been coming out to improve both theory and practice. Many are the contributors who left their mark on the field, among them Sharpe (1964), Tobin (1958), Lintner (1965), Fama (1970), Mossin (1968), Merton (1990), Ross (1976), Roll (1978), Jensen (1968), Blume (1971), Elton and Gruber (1995). All of them, and many others, brought about theoretical breakthroughs, analitical insights or powerful empirical tools so as to enhance models in their trading with real world problems. A well-chosen survey on this development can be found in Elton-Gruber (1997).

This paper will address arbitrage portfolios, which are set up to take profit of mispriced securities or portfolios against another security or portfolio that seems correctly priced in the context of some valuation model that might have been taken as a benchmark. Furthermore, we are going to handle two benchmarks supplied by the CAPM: the capital market line (CML) and the security market line (SML).

It should be expected from this paper an expansion on some distinctive issues regarding arbitrage portfolios.

(a) A definition on arbitrage portfolios that enables adjustments to SML and CML environments.

(b) Sufficient conditions to set up arbitrage portfolios against the SML and CML.

(c) Feasibility of separation portfolios to carry out arbitrage not only against SML but CML as well.

(d) Arbritrage of portfolios located in Treynor lines by using separation portfolios within a SML environment.

To develop these issues the following steps will be taken:

- In section 1 arbitrage portfolios are introduced. Then, the market model, the CML and the SML environments are briefly surveyed.

- Section 5 deals with the bridge between the CML and the SML as it comes by linking betas and sigmas on the one hand, and null residual variances on the other hand.

- Next, Lemma 1 will establish a constructive way to arbitrage in the SML environment, whereas Lemmas 2 and 3 cope with the same issue but with the CML as benchmark.

- Section 8 brings a Lemma that establishes feasibility conditions for separation portfolios to be used in arbitrage when SML is the benchmark. Section 9 labours a similar outcome but here the separation portfolios are matched against portfolios in Treynor Lines.

## 1.- ARBITRAGE PORTFOLIOS

As from now, we are going to deal with the set of all portfolios **DP** that exhibit the following features as regards valuation setting, portfolio structure and boundary conditions.

*a) Valuation setting*

i. Horizon: [ t ; T ]
ii. Information set: $\Omega_t$
iii. Valuation date: t

*b) Structure*

i. It is comprised of two portfolios or securities A and B.
ii. Whenever one of them is held as a long position, the other one is held as a short position.

*c) Boundary conditions*

The whole proceeds from the short position are used up for purchasing the long position (self-financing). That is to say, if x(A), x(B), stand for the participations of A and B, respectively, into the portfolio **DP**, we can write:

**[ 01 ]**

$$\mathbf{DP} = \langle x(A) ; x(B) \rangle = \begin{cases} \langle +1 ; -1 \rangle & \text{when A is held long} \\ \langle -1 ; +1 \rangle & \text{when A is held short} \end{cases}$$

Among these portfolios **DP** there are some that are worthy of interest on their own: the so-called arbitrage portfolios.

***Definition 1***

*A portfolio **DP** will be called an arbitrage portfolio at the moment " t " if the following conditions are met:*

*i) For the same relevant measure of risk, it holds true that:*

*relevant measure of risk for **LP** = relevant measure of risk for **SP***

*ii) The expected return of portfolio **DP** is strictly positive:*

$$E[\,R\,(DP)\,] > 0$$

In other words, arbitrage portfolios are self-financing, riskless and profitable.

**Remarks:**

(a) The customary notions of arbitrage portfolios take into account the riskless, self-financing and profitable features that are brought about, at a given moment " t ". This means that arbitrage portfolios are usually settled in the spot market.

(b) In doing so, the usual definition neglects not only transaction costs, but the forward risky position they leave open as well. A transaction costs approach to financial assets can be found in Apreda (2000a ; 2000b).

(c) On the other hand, we should bear in mind that, although we simplify notation, the profit condition stated by ii) in the definition, it really stands for:

$$E[\,R\,(\Delta P)\,] \;=\; E[\,R\,(\Delta P) \mid \Omega_t\,] \;>\; 0$$

that is to say, the expected operator is restricted to the information set $\Omega_t$ at date " t ". The role of information sets in computing securities rates of return is handled in Apreda (2000c).

## 2.- THE MARKET MODEL

This is a model to forecast correlation structures of securities returns. An alternative intuition tells that it conveys a return-generating process. A current formal setting for the market model turns out to be the following:

$$[02] \quad \begin{cases} R(K) \;=\; a(K) \;+\; b(K)\,.\,R(M) \;+\; e(K) & ; \quad K : 1, 2, \dots, N \\ E[\,e(K)\,] \;=\; 0 & ; \quad K : 1, 2, \dots, N \\ Cov\,(\,K\,;\,M\,) \;=\; E[\;\{\,e(K) - E[\,e(K)\,]\,\}\,.\,\{\,R(M) - E[\,R(M)\,]\,\}\;] \;=\; 0 & ; \quad K : 1, 2, \dots, N \end{cases}$$

**Remarks:**

(a) A state of the art development on the market model can be found in Elton-Gruber (1995).

(b) It is supposed that there are N securities in the market at the date of valuation and that N does not change along the valuation horizon [t ; T].

(c) Sometimes it is added another assumption

$$Cov\,(\,e(\,K\,)\,;\,e(\,J\,)\,) = E[\;\{\,e(\,K\,) - E[\,e(\,K\,)\,]\,\}\,.\,\{\,e(\,J\,) - E[\,e(\,J\,)\,]\,\}\;] = 0 \;;\; K : 1, 2, \dots, N$$

which sets forth that there is no industry effects. Nevertheless, it has been rejected most of the time by empirical work. It is interesting to remember that Fama(1968) corrected a discrepancy between Sharpe(1964) and Lintner (1965) on the ground that this assumption was inconsistent. Later Beja(1972) pointed at some inaccuracies in Fama's remarks and, finally, Fama(1973) redressed his former paper adding more precision to settle this issue. Blume(1971) is also useful, in particular when compares the portfolio approach with the equilibrium approach.

There are two main outcomes that stem from the market model, the first one concerning single securities, the second one focusing on portfolios. For any single asset K, it holds true that:

$$E[\,R(K)\,] \;=\; \alpha(K) \;+\; \beta(K) \,.\, E[\,R(M)\,]$$

$$\sigma^2(K) \;=\; \beta(K) \,.\, \sigma^2(M) \;+\; \sigma^2(\varepsilon(K))$$

where $\sigma^2(\varepsilon(K))$ stands for each asset K's residual variance, which measures specific or non systematic risk. The extension to portfolios

$$P \;=\; < x(1) \,;\, x(2) \,;\, x(3) \,;\, \ldots\ldots \,;\, x(N) >$$

where x(j) means the fractional allocation of wealth to the jth asset, it follows outright.

**[03]**

$$\begin{cases} E[\,R(P)\,] \;=\; \alpha(P) \;+\; \beta(P) \,.\, E[\,R(M)\,] \\ \sigma^2(P) \;=\; \beta(P) \,.\, \sigma^2(M) \;+\; \sigma^2(\varepsilon(P)) \end{cases}$$

such that

$$\alpha(P) \;=\; \sum_{1 \leq k \leq N} x(k) \,.\, \alpha(k) \quad ; \quad \beta(P) \;=\; \sum_{1 \leq k \leq N} x(k) \,.\, \beta(k)$$

$$\sigma^2(\varepsilon(P)) \;=\; \sum_{1 \leq k \leq N} x^2(k) \,.\, \sigma^2(\varepsilon(k))$$

It has been argued that a linear market model is a sufficient condition for a linear relationship between betas and expected returns, as conveyed by the SML (Stapleton and Subrahmantam, 1983). Such keynote statement allows the application of a statistic model (the market model) within the context of an economic model (the CAPM) as we are going to do in sections to come after.

## 3.- THE CAPITAL MARKET LINE

Let us suppose that P is a portfolio of separation, that is to say

$$P \;=\; < x(F) \;;\; x(M) >$$

which is made out of the risk-free asset and the market portfolio. In practice, a market proxy is used, like the NYSE index or SP500 (Garbade, 1982).

Computing the expected return and the variance for this portfolio, we get:

$$\begin{cases} E[\,R(P)\,] \;=\; x(F) \,.\, E[\,R(F)\,] \;+\; x(M) \,.\, E[\,R(M)\,] \;=\; x(F) \,.\, E[\,R(F)\,] \;+\; x(M) \,.\, E[\,R(M)\,] \\ \sigma^2(P) \;=\; x^2(F) \,.\, \sigma^2(F) \;+\; x^2(M) \,.\, \sigma^2(M) \;+\; 2 \,.\, x(F) \,.\, x(M) \,.\, \sigma(F,M) \end{cases}$$

As the risk-free asset has null variance by itself, and null covariance with the market portfolio, it holds

$$\sigma^2(P) = x^2(M) \, . \, \sigma^2(M) \quad \Rightarrow \quad \sigma(P) = x(M) \, . \, \sigma(M)$$

that leads to

**[04]** $$x(M) = \sigma(P) \; / \; \sigma(M)$$

which can be substituted in the expected return of the separation portfolio and soattain:

**[05]** $$E[\,R(P)\,] = R(F) + \{ \; ( \; E[\,R(M)\,] - R(F) \; ) \; / \; \sigma(M) \; \} \times \sigma(P)$$

This is a remarkable outcome, because it links the expected return with its standard deviation by means of a linear relationship, usually known as the Transformation Line (Cuthbertson, 1996). Furthermore, endowed with CAPM assumptions it becomes the Capital Market Line, that is to say, the CML (Sharpe, 1964).

## 4.- THE SECURITY MARKET LINE

Let us denote with K any single asset or any portfolio. In equilibrium, its expected return turns out to be

$$E[\,R(K)\,] = R(F) + ( \; E[\,R(M)\,] - R(F) \; ) \; . \; \beta(K)$$

**Remark**

Hence, when the broad picture conveyed by the CAPM is further developed, both linear relationships seems instrumental for a better understanding of capital markets. It is for the CML to describe the efficient frontier consisting only of separation portfolios. In turn, the SML performs as a useful device to assessing the price of financial assets whenever equilibrium holds, in terms of systematic risk only.

By remembering that

$$\beta(K) = \mathrm{cov}\,(\,K\,;\,M\,) \; / \; \sigma^2(M)$$

a distinctive linkage between the CML and SML can be underlined further. Firstly, the SML becomes

$$E[\,R(K)\,] = R(F) + \{ \; ( \, E[\,R(M)\,] - R(F) \; ) \, / \; \sigma(M) \; \} \, . \, \{ \, \mathrm{cov}\,(\,K\,;\,M\,) \; / \; \sigma(M) \, \}$$

Secondly, if we contrast this relationship with the CML:

$$E[\,R(P)\,] = R(F) + \{ \; ( \, E[\,R(M)\,] - R(F) \; ) \; / \; \sigma(M) \, \} \, . \, \sigma(P)$$

we realize that, whereas the CML adjusts the risk of a separation portfolio by means of the price-of-risk and the absolute risk of the portfolio, the SML adjusts the risk of any single asset or portfolio by means of the price-of-risk and the marginal risk the market portfolio bears on when adding a unit of asset K. In fact, marginal risk comes down to to the marginal contribution of the Kth security to portfolio risk:

$$\partial \sigma(M) / \partial x(K) = cov(K ; M) / \sigma(M)$$

**Remarks:**

(a) Full derivation of this outcome, as well expansions on related issues, can be found in Elton-Gruber (1995).

(b) In equilibrium, $( E[ R(M) ] - R(F) ) / \sigma(M)$ comes out of two marginal rates being equal: the substitution rate between risk and return for investors and the rate of transformation between risk and return in the CML. Blake (1999) provides a good account of this issue.

## 5.- THE BRIDGE BETWEEN THE CML AND THE SML

Whenever we take a portfolio P $\in$ CML, then it also fulfills the Security Market Line relationship because, in equilibrium, any portfolio or single asset lies on the SML.

$$E[ R(P) ] = R(F) + ( E[ R(M) ] - R(F) ) . \beta(P)$$

But $\beta(P)$ is given by

$$\beta(P) = cov( R(P) ; R(M) ) / \sigma^2(M)$$

On the other hand:

$$cov( R(P) ; R(M) ) = E[ ( R(P) - E[ R(P) ] ) . ( R(M) - E[ R(M) ] ) ]$$

$$cov( R(P) ; R(M) ) = E[ x(M) . ( R(M) - E[ R(M) ] ) . ( R(M) - E[ R(M) ] ) ]$$

$$cov( R(P) ; R(M) ) = x(M) . E[ ( R(M) - E[ R(M) ] ) . ( R(M) - E[ R(M) ] ) ]$$

$$cov( R(P) ; R(M) ) = x(M) . \sigma^2(M)$$

Therefore:

$$\beta(P) = cov( R(P) ; R(M) ) / \sigma^2(M)$$

**[06]**

$$\beta(P) = x(M)$$

Therefore, separation portfolios carry on this feature: their participation of market portfolio translates into their betas. By [04] and [06] ,

$$\beta(P) = x(M) = \sigma(P) / \sigma(M)$$

and we get, at last:

**[07]**

$$\sigma(P) = \beta(P) . \sigma(M)$$

which allows us to deal with the total risk of the portfolio by means of its beta and the total risk of the market. As we see, it is for [07] to cross the bridge from SML to CML when we deal with separation portfolios.

Let us further assume that we choose a portfolio P ∈ CML which has as coordinates in the SML world

$$< b(P) \; ; E[\, R(P) \,] >$$

This portfolio becomes specified on the SML by [03] and it has the following variance:

$$s^{2}(P) \;=\; b^{2}(P)\;.\;s^{2}(M) \;+\; \textbf{residual variance(P)}$$

On the other hand, as it is an efficient portfolio in the CML, we can use [04]

$$s^{2}(P) = x^{2}(M)\,.\,s^{2}(M)$$

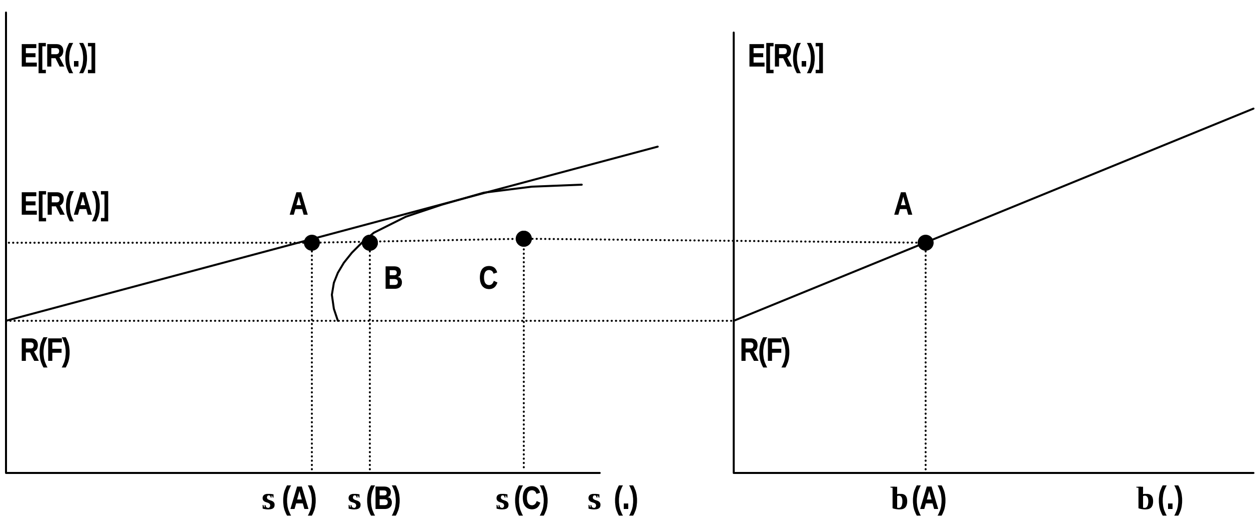
Taking advantage of [06], [07], and the last two variance expansions, it follows

$$s^{2}(P) \;=\; b^{2}(P)\,.\,s^{2}(M) \;=\; b^{2}(P)\;.\;s^{2}(M) \;+\; \textbf{residual variance(P)}$$

hence:

$$\textbf{residual variance(P)} \;=\; 0$$

Next picture is worthy of remark:




(a) As portfolio A ∈ CML , it holds, by [04] and [07],

$$s(A) \;=\; x(M)\,.\,s(M) \;=\; b\,(A)\,.\,s\,(M)$$

(b) B is a Markovian portfolio, not efficient within the CML world. By applying the market model we get a qualification of risk for B:

$$s^{2}(B) \;=\; b^{2}(B)\,.\,s^{2}(M) \;+\; \textbf{residual variance(P)}$$

But

$$E[\,R(A)\,] \;=\; E[\,R(B)\,]$$

and solving in the SML world:

$$\beta\,(A) \;=\; \beta\,(B)$$

(c) By the same token,

$$\beta\,(A) \;=\; \beta\,(P)$$

for any portfolio P in the region of feasible portfolios (in Markowitz's sense), and lying on the level set

$$L[\,A\,] \;=\; \{\; P \;:\; A \in CML \;;\; E[\,R(A)\,] \;=\; E[\,R(P)\,]\;\}$$

(d) If all portfolios lying on L[A] share the same beta, this means,

$$Cov(\,R(P)\,;R(M)\,) \;=\; Cov(\,R(A)\,;R(M)\,)$$

## 6.- ARBITRAGE PORTFOLIOS WITH THE SECURITY MARKET LINE

We are going to set up arbitrage portfolios using the SML as benchmark and, in order to accomplish it, we need to narrow the meaning of any arbitrage portfolio as depicted in Definition 1.

***Definition 2***

*It is said that* $\Delta P$ *is an arbitrage portfolio in the environment provided by the SML if it fulfills:*

$$\begin{cases} \Delta P \;=\; <\, x(A)\,;x(B)\,> \;=\; \begin{cases} <\,+1\,;-1\,> & \text{when A is held long} \\ <\,-1\,;\,+1\,> & \text{when A is held short} \end{cases} \\ A \notin SML \;,\; B \in SML \\ \beta(\Delta P) \;=\; 0 \\ E[\,R(\Delta P)\,] = \; x(A)\,.\,E[\,R(A)\,] \;+\; x(B)\,.\,E[\,R(B)\,] \;>\; 0 \end{cases}$$

Could $\Delta P$ belong to the SML world? Never. Firstly because, in such case, as its beta equals zero, it would follow:

$$E[\,R(\Delta P)\,] \;=\; R(F) \;+\; \{\; E[\,R(M)\,] - R(F)\;\}\,.\,\beta(\Delta P)$$

$$E[\,R(\Delta P)\,] \;=\; R(F)$$

and this is unlikely to happen. Secondly, **DP** couldn't belong to SML because it comes out of disequilibrium, consisting of a mispriced security or portfolio A. and a well priced portfolio B. Thirdly, if **DP** belonged to SML, then E[ R(**DP**) ] should equal zero.

What sort of portfolios might be arbitraged against the SML? From a theoretical point of view, all mispriced portfolios or securities could, for certain. On empirical grounds, however, there are distinctive hindrances to perform successful arbitrages. On this account, Shleifer (1999) deals with inefficient markets preventing arbitrage to be carried out. It is for the following lemma, however, to warrant feasibility to the theoretical arbitrage.

***Lemma 1:***

***Any mispriced portfolio A can be arbitraged against the SML. Therefore, arbitrage portfolios are feasible in the context of SML.***

Proof: Let us distinguish two cases: in the first one, there is a single accessible portfolio B ∈ SML to match against A; in the second one, we have to synthesize B by means of two accessible portfolios C and D, both lying on the SML.

a) If we found a portfolio A whose expected return is greater than what it is predicted by the SML

$$E[\,R(A)\,] \;>\; R(F) \;+\; \{\,E[\,R(M)\,] - R(F)\,\}\,.\,b\,(A)$$

then A would not belong to the SML (A ∉ SML). By choosing a portfolio B ∈ SML with the same beta as A, we could sell B and with the proceedings buy A. The whole process, which involves a sale and a purchase at the same time, brings about a new portfolio **DP** with distinctive features on its structure, initial investment and expected returns.

$$\begin{cases} DP \;=\; <x(A)\,;\,x(B)> \;=\; <+1\,;\,-1> \\ A \;\; Ï \;\; SML\;,\; B\; Î \; SML \\ b\,(A) \;=\; b\,(B) \\ b\,(DP) \;=\; x(A)\,.\,b\,(A) \;+\; x(B)\,.\,b\,(B) \;=\; 0 \end{cases}$$

The initial wealth of this portfolio is zero. In fact, when selling we get W dollars which we use outright to purchase the portfolio A :

$$x(A)\,.\,W \;+\; x(B)\,.\,W \;=\; W \;-\; W \;=\; 0$$

As we see, portfolio **DP** is selfinancing.

By construction, it follows that $E[\,R(A)\,] \;>\; E[\,R(B)\,] \;=\; R(F) \;+\; \{\,E[\,R(M)\,] - R(F)\,\}\,.\,b\,(A)$

then,

$$E[\,R(\Delta P)\,] \;=\; x(A)\,.\,E[\,R(A)\,] \;+\; x(B)\,.\,E[\,R(B)\,]$$

$$E[\,R(\Delta P)\,] \;=\; E[\,R(A)\,] \;-\; E[\,R(B)\,] \;>\; 0$$

Assuming the expected return of portfolio A were less than the value it would earn in the SML, we can carry out a similar kind of analysis. In this setting:

$$E[\,R(B)\,] \;>\; E[\,R(A)\,]$$

$$E[\,R(\Delta P)\,] \;=\; x(A)\,.\,E[\,R(A)\,] \;+\; x(B)\,.\,E[\,R(B)\,]$$

and as

$$\Delta P \;=\; < x(A)\,;\, x(B) > \;=\; < -1\,;\, +1 >$$

it follows that

$$E[\,R(\Delta P)\,] \;>\; 0$$

b) Let us suppose now that portfolio B is not accessible outright (for instance, because short selling is forbidden, or there are shortages in some of the portfolio components). We could package up two accessible and well known portfolios C, D, such that:

$$[08]\quad \begin{cases} C, D \in SML \quad ; \quad B = < x(C)\,;\, x(D) > \\ x(C)\,.\,W(C) + x(D)\,.\,W(D) = W \quad \text{or, equivalently,} \quad x(C) + x(D) = 1 \\ \beta(B) = x(C)\,.\,\beta(C) + x(D)\,.\,\beta(D) = \beta(A) \end{cases}$$

We need to solve for **x(C)**, **x(D)** in [08]

$$\beta(A) \;=\; x(C)\,.\,\beta(C) + x(D)\,.\,\beta(D) \;=\; x(C)\,.\,\beta(C) + (1 - x(C))\,.\,\beta(D)$$

therefore:

$$x(C) = \{\beta(A) - \beta(D)\} \;/\; \{\beta(C) - \beta(D)\}$$

$$x(D) = \{\beta(C) - \beta(A)\} \;/\; \{\beta(C) - \beta(D)\}$$

hence,

$$\Delta P \;=\; < x(A)\,;\, x(B) > \;=\; < x(A)\,;\, < x(C)\,;\, x(D) > >$$

is an arbitrage portfolio. Š

## 7.- ARBITRAGE PORTFOLIOS WITH THE CAPITAL MARKET LINE

What does it mean an arbitrage portfolio when we move into the context of the CML? The answer is provided by the following definition.

***Definition 3***

*It is said that* $\Delta P$ *is an arbitrage portfolio in the environment provided by the CML if it fulfills:*

$$\begin{cases} \Delta P = \langle x(A) \,;\, x(B) \rangle = \begin{cases} \langle +1 \,;\, -1 \rangle & \text{when A is held long} \\ \langle -1 \,;\, +1 \rangle & \text{when A is held short} \end{cases} \\ A \notin CML \;,\; C \in CML \\ \sigma(\Delta P) = 0 \\ E[\,R(\Delta P)\,] = x(A)\,.\,E[\,R(A)\,] + x(B)\,.\,E[\,R(B)\,] > 0 \end{cases}$$

It seems worthy of interest to find out whether we can arbitrage portfolios against the CML the way we did in the SML context. Whereas all single assets and portfolios lie on the SML, only separation portfolios locate in the CML. That is why we are going to deal with the following pair of environments:

(a) Portfolio A is a separation one, although temporarily mispriced.

(b) Portfolio A is inefficient, although temporarily mispriced, and lies above or below the CML.

In each environment we are going to prove a Lemma that gives a sufficient condition for an arbitrage portfolio to be feasible or not.

***Environment 1: The portfolio A is a separation portfolio which is temporarily mispriced.***

***Lemma 2:***

***If A*** $\notin$ ***CML but it is a temporarily mispriced separation portfolio, then a portfolio arbitrage against the CML is feasible.***

Proof: For instance, let us assume that it is cheaper than it would be if it lay in the CML. In other words, it lies above the CML. (a similar expansion would follow if we suppose the portfolio were more expensive, lying below the CML).

Now, we proceed to build up a portfolio with the following features:

$$\begin{cases} \mathbf{DP} = \langle x(A)\ ;\ x(B) \rangle = \langle \delta(A)\ ;\ \delta(B) \rangle = \langle +1\ ;\ -1 \rangle \\ \sigma(B) = \sigma(A) \\ x(A) . W + x(B) . W = W - W = 0 \end{cases}$$

We want to work out both the expected return and variance of portfolio **DP**.

$$E[\,R(DP)\,] = x(A) . E[\,R(A)\,] + x(B) . E[\,R(B)\,]$$

$$E[\,R(DP)\,] = E[\,R(A)\,] - E[\,R(B)\,] > 0$$

As far as the variance is concerned, we have:

$$\sigma^2(DP) = x^2(A) . \sigma^2(A) + x^2(B) . \sigma^2(B) + 2 . x(A) . x(B) . \sigma(A,B)$$

**[09]** $$\sigma^2(DP) = \sigma^2(A) + \sigma^2(B) - 2\,\sigma(A,B)$$

On the other hand,

$$\sigma(A) = x(M;A) . \sigma(M) = \beta(A) . \sigma(M)$$

$$\sigma(B) = x(M;B) . \sigma(M) = \beta(B) . \sigma(M)$$

$$\sigma(A,B) = x(M;A) . x(M;B) . \sigma^2(M) = \beta(A) . \beta(B) . \sigma^2(M)$$

Replacing in [09] and bearing in mind that

$$\sigma(B) = \sigma(A) \quad \Rightarrow \quad \beta(A) = \beta(B)$$

it follows that

$$\sigma^2(DP) = \beta^2(A) . \sigma^2(M) + \beta^2(B) . \sigma^2(M) - 2\,\beta(A) . \beta(B) . \sigma^2(M) = 0$$

thus, since definition 3, **DP** becomes an arbitrage portfolio. By the same token, we could have dealt with a short position in A. ■

***Environment 2: Let us assume that an inefficient portfolio A lies above (below) the CML.***

***Lemma 3:***

***If A $\notin$ CML and it is not a separation portfolio, then a portfolio arbitrage against the CML is not feasible.***

Proof: We want to try an arbitrage portfolio by means of A against the CML. By solving

$$\begin{cases} E[\,R(B)\,] = R(F) + \{\,(\,E[\,R(M)\,] - R(F)\,)\ /\ \sigma(M)\,\}\,.\,\sigma(B) \\ \sigma(B) = \sigma(A) \end{cases}$$

we can determine a portfolio $B \in$ CML such that

$$B = <x(F)\,;\,x(M, B)>$$

which fulfills [05] and it can be matched with portfolio A. In fact, if A lay above the CML it would follow

$$E[\,R(A)\,] > E[\,R(B)\,]$$

We set up a portfolio **ΔP** whose structure comes from

$$[10] \qquad \begin{cases} \Delta P = <x(A)\,;\,x(B)> = <+1\,;\,-1> \\ A \notin CML \quad ; \quad B \in CML \\ \sigma(B) = \sigma(A) \\ x(A)\,.\,W + x(B)\,.\,W = W - W = 0 \end{cases}$$

These conditions warrant that there could be a benchmark, namely the CML against which we could arbitrage eventually if both risk and return of **ΔP** fulfilled definition 3. Besides, it is self-financing.

Let us try to figure out the return-risk profile of portfolio **ΔP**. Firstly, its expected return comes out of:

$$E[\,R(\Delta P)\,] = x(A)\,.\,E[\,R(A)\,] + x(B)\,.\,E[\,R(B)\,]$$

$$E[\,R(\Delta P)\,] = E[\,R(A)\,] - E[\,R(B)\,] > 0$$

Secondly, the portfolio **ΔP** variance follows from:

$$\sigma^2(\Delta P) = x^2(A)\,.\,\sigma^2(A) + x^2(B)\,.\,\sigma^2(B) + 2\,.\,x(A)\,.\,x(B)\,.\,\sigma(A,B)$$

and taking advantage of portfolio **ΔP** structure in [10],

$$\sigma^2(\Delta P) = \sigma^2(A) + \sigma^2(B) - 2\,.\,\sigma(A,B)$$

Furthermore,

$$\sigma(A,B) = E[\,<R(A) - E[\,R(A)\,]>\,.\,<R(B) - E[\,R(B)\,]>\,]$$

Bearing in mind that A $\notin$ CML but B $\in$ CML, it comes next that

$$R(A) - E[\,R(A)\,] \;=\; \beta(A)\,.\,\{\,R(M) - E[\,R(M)\,]\,\} \;+\; \varepsilon(A)$$

$$R(B) - E[\,R(B)\,] \;=\; x(M, B)\,.\,\{\,R(M) - E[\,R(M)\,]\,\}$$

hence:

$$\sigma\,(A,B) \;=$$

$$= \beta(A)\,.\,x(M, B)\,.\,E[\,\{\,R(M) - E[\,R(M)\,]\,\}\,]^{2} + x(M, B)\,.\,E[\,\{\,R(M) - E[\,R(M)\,]\,\}.\{\,\varepsilon(A) - E[\,\varepsilon(A)]\,\}$$

that is to say, by using market model assumptions:

$$\sigma\,(A,B) \;=\; \beta(A)\,.\,x(M, B)\,.\,\sigma^{2}(M)$$

Therefore:

$$\sigma^{2}(DP) \;=\; \sigma^{2}(A) \;+\; \sigma^{2}(B) \;-\; 2\,.\,\sigma\,(A,B)$$

$$\sigma^{2}(DP) \;=\; \sigma^{2}(A) \;+\; \sigma^{2}(B) \;-\; 2\,.\,\beta(A)\,.\,x(M, B)\,.\,\sigma^{2}(M)$$

On the other hand:

$$\sigma\,(B) \;=\; x(M, B)\,.\,\sigma\,(M) \;=\; \beta(B)\,.\,\sigma\,(M)$$

and portfolio A, by the market model has the following variance:

$$\sigma^{2}(A) \;=\; \beta^{2}(B)\,.\,\sigma^{2}(M) \;+\; \sigma^{2}(\varepsilon(A))$$

Thus, we are led to:

$$\sigma^{2}(DP) \;=\; \sigma^{2}(A) + \sigma^{2}(B) \;-\; 2\,.\,\beta(A)\,.\,x(M, B)\,.\,\sigma^{2}(M)$$

$$\sigma^{2}(DP) \;=\; \beta^{2}(B)\,.\,\sigma^{2}(M) \;+\; \sigma^{2}(\varepsilon(A)) \;+\; \beta^{2}(B)\,.\,\sigma^{2}(M) \;-\; 2\,.\,\beta(A)\,.\,\beta(B)\,.\,\sigma^{2}(M)$$

rearranging

$$\sigma^{2}(DP) \;=\; <\beta^{2}(A) + \beta^{2}(B) - 2\,.\,\beta(A)\,.\,\beta(B)>\,.\,\sigma^{2}(M) \;+\; \sigma^{2}(\varepsilon(A))$$

As A is not a separation portfolio, it follows that

**[11]** $$\sigma^{2}(DP) \;=\; <\beta\,(A) - \beta\,(B)>^{2}\,.\,\sigma^{2}(M) \;+\; \sigma^{2}(\varepsilon(A)) \;\neq\; 0$$

Then, **DP** does not become an arbitrage portfolio. By the same token, we could have reached a similar outcome if we had chosen a short position in A ∎

## 8.- ARBITRAGING WITH SEPARATION PORTFOLIOS IN THE SML CONTEXT

It has been settled that arbitrage portfolios are always feasible (Lemma 1), at last from a theoretical perspective. It should convey practical advantages if we could arbitrage against the SML with separation portfolios outright as we did in the context of CML (Lemma 2 and Lemma 3). After all, separation portfolios are located in the SML. This issue is addressed by next Lemma.

***Lemma 4:***

***If A is any single risky asset or portfolio of risky assets which is mispriced against the SML, then it holds true that:***

***(a) In the context of SML, there is a separation portfolio B in SML that matches A in an arbitrage portfolio.***

***(b) In the context of CML, portfolio B can not match A in an arbitrage portfolio.***

***(c) Portfolio B becomes functional to determine an effective separation portfolio C that matches the absolute risk of A, in the CML context.***

Proof: (a) taking a portfolio A whose coordinates in the SML context are

$$\langle \beta(A) ; E[R(A)] \rangle$$

we proceed to single out as B another portfolio with the following features:

i. $B \in$ SML
ii. B is a separation portfolio
iii. $\beta(A) = \beta(B)$

As B is a separation portfolio,

$$B = \langle x(F) ; x(M) \rangle$$

and

$$x(M) = \beta(B)$$

Therefore, in order to arbitrage A we have to choose $x(M) = \beta(A)$.

(b) As B also belongs to the CML, it holds that

**[12]** $$\sigma(B) = \beta(B) \cdot \sigma(M)$$

On the other hand, as both portfolios have the same beta, by the market model:

$$\sigma^2(A) = \beta^2(B) \cdot \sigma^2(M) + \sigma^2(\varepsilon(A))$$

and by [12]

$$\sigma^2(A) = \sigma^2(B) + \sigma^2(\varepsilon(A))$$

Therefore:

$$\sigma(A) > \sigma(B)$$

c) Finally, let us choose the portfolio C such that:

i. $C \in CML$
ii. $\sigma(A) = \sigma(C)$

Then, by [04]

$$\sigma(C) = x(M, C) \,.\, \sigma(M)$$

and

$$[13] \qquad \sigma^2(A) = x^2(M, B) \,.\, \sigma^2(M) + \sigma^2(\varepsilon(A)) = \sigma^2(C) = x^2(M, C) \,.\, \sigma^2(M)$$

Therefore, we can solve for x(M, C) in [12] and get

$$x(M, C) = \{ x^2(M, B) + [ \sigma^2(\varepsilon(A)) / \sigma^2(M) ] \}^{1/2}$$

then, the portfolio < x(F) ; x(M, C) > is risk equivalent to A, but $\sigma(\Delta P) \neq 0$, as we saw in Lemma 3. ∎

## 9.- ARBITRAGING PORTFOLIOS BETWEEN THE SML AND TREYNOR LINES.

In the context of the CML, separation portfolios are defined as

$$P = < x(F) ; x(M) >$$

which can be thought as portfolios that come out of two trust portfolios: one with strictly risky assets and the other one with only riskless assets; that is to say, the market portfolio and the risk-free asset.

If we move now to the SML, we could build up separated mutual funds portfolios whose structure comes out of a portfolio Q of strictly risky assets, which is mispriced against the SML, and a risk-free asset or portfolio. That is to say, Q and F span the set of all portfolios

$$A = < x(F) ; x(Q) >$$

The place where all these portfolios lie is a line whose derivation follows that of the capital allocation line [05], almost word by word. But the key point to bear in mind is that whereas the capital allocation line uses an absolute measure of risk, $\sigma$, the new line uses a relative measure of risk, beta. In fact, the lines where these portfolios lie follow the pattern:

$$E[ R(A) ] = R(F) + ( \{ E[ R(Q) ] - R(F) \} / \beta(Q) ) \,.\, \beta(A)$$

The slope ot this line is nothing else but the Treynor Index, whereof the convenience of calling it the Treynor Line.

***Lemma 5:***

***Let A be any portfolio such that:***

$$A = < x(F) ; x(Q) >$$

***where Q is a risky portfolio which is mispriced against the SML. Then the following statements hold true:***

***(a) Portfolio A can be matched with another one lying on the SML in order to set up an arbitrage portfolio.***

***(b) There is a separation portfolio belonging to the SML environment which depends only on Q and matches A in a arbitrage portfolio .***

Proof: (a) let us pick up any risky portfolio Q and frame the set of all portfolios A with the following structure:

$$A = < x(F) ; x(Q) >$$

That is to say, we set up portfolios that stem from participation units in a trust portfolio of strictly risky assets and participation units in a risk-free fund. In the context of the market model

$$b\,(A) = x(F) . b\,(F) + x(Q) . b\,(Q)$$

**[14]** $$b\,(A) = x(Q) . b\,(Q)$$

and solving for x(Q):

$$x(Q) = b\,(A) \,/\, b\,(Q)$$

Next, working out the expected return of portfolio A,

$$E[\,R\,(A)\,] = x(F) . R(F) + x(Q) . E[\,R(Q)\,]$$

$$E[\,R\,(A)\,] = (\,1 - b\,(A) \,/\, b\,(Q)\,) . R(F) + b\,(A) \,/\, b\,(Q) . E[\,R(Q)\,]$$

and rearranging

**[15]** $$E[\,R\,(A)\,] = R(F) + (\,\{\,E[\,R(Q)\,] - R(F)\,\} \,/\, b\,(Q)\,) . b\,(A)$$

As we see, risk adjustment is provided by the Treynor Index:

$$( \{ E[ R(Q) ] - R(F) \} \ / \ b \ (Q) \ )$$

and only if all CAPM assumptions hold, then [14] becomes the SML.

Now, we want to arbitrage any portfolio

$$A = < x(F) \ ; x(Q) >$$

against the SML. We have to choose a portfolio B such that:

i. $B \in SML$
ii. $\beta(B) = \beta(A)$

Let us suppose that it holds that

$$E[ R(A) ] > E[ R(B) ]$$

this translates into

$$R(F) + ( \{ E[ R(Q) ] - R(F) \} / b \ (Q) \ ) . b \ (A) \ ) \quad > \quad R(F) + \{ E[ R(M) ] - R(F) \} . b \ (B)$$

Taking advantage of the fact that betas are equal:

**[16]** $$( \{ E[ R(Q) ] - R(F) \} / b \ (Q) \ ) \quad > \quad \{ E[ R(M) ] - R(F) \}$$

On the other hand, if it held

$$E[ R(A) ] < E[ R(B) ]$$

we would end up with:

**[17]** $$( \{ E[ R(Q) ] - R(F) \} \ / \ b \ (Q) \ ) \quad < \quad \{ E[ R(M) ] - R(F) \}$$

Lastly, if we choose x(Q) as the desired allocation of wealth for portfolio Q, then portfolio A would have as beta

$$b(A) = x(Q) . b(Q)$$

and this will supply the beta of portfolio B.

(b) By Lemma 5, we can always arbitrage portfolio A with a separation portfolio $B \in SML$. On the other hand, if A belonged to a Treynor line, then

$$A = < x(F,A) \ ; x(Q) > \ = \ < 1 - x(Q) \ ; x(Q) >$$

$$B = < x(F,B) \ ; x(M,B) > \ = \ < 1 - x(M,B) \ ; x(M,B) >$$

by [14] and [14], taking advantage of Lemma 5 we get:

$$\beta(A) = x(Q) \,.\, \beta(Q)$$

$$\beta(B) = x(M,B)$$

and we only have to choose:

$$x(M,B) = x(Q) \,.\, \beta(Q)$$

and the arbitrage portfolio comes out of:

$$\Delta P = < x(A) \; ; \; x(B) > = < +1 \, ; -1 >$$

with the following structure:

$$\Delta P = \{ < x(F,A) \; ; \; x(Q) > ; < x(F,B) \; ; \; x(M,B) > \}$$

$$\Delta P = \{ < 1 - x(Q) \; ; \; x(Q) > ; < 1 - x(M,B) \; ; \; x(M,B) > \}$$

$$\Delta P = \{ < 1 - x(Q) \; ; \; x(Q) > ; < 1 - x(Q) \,.\, \beta(Q) \; ; x(Q) \,.\, \beta(Q) > \}$$ Š

## CONCLUSIONS

By properly defining arbitrage portfolios, we can set up firstly a user friendly environment and, secondly, sufficient conditions dealing with arbitrage portfolios against the SML and the CML. The role of separation portfolios has been enhanced, because under certain assumptions they allow for arbitraging against the Capital Market Line. Finally, we showed how to use separation portfolios in the world of the Security Market Line, so as to take advantage of mispriced portfolios lying on Treynor lines.

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