NBER WORKING PAPER SERIES

LONG-TERM DEBT AND OPTIMAL POLICY IN THE FISCAL THEORY OF THE PRICE LEVEL

John H. Cochrane

Working Paper 6771
http://www.nber.org/papers/w6771

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 1998

I thank the CRSP, Graduate School of Business and the National Science Foundation for research support, and I thank Andrea Eisfeldt for research assistance. An early draft of this paper circulated under the title ``Maturity Matters: Long Term Debt in the Fiscal Theory of the Price Level." The views expressed here are those of the author and do not reflect those of the National Bureau of Economic Research.

Long-term Debt and Optimal Policy
in the Fiscal Theory of the Price Level
John H. Cochrane
NBER Working Paper No. 6771
October 1998
JEL No. E3, E4, E5

ABSTRACT

The fiscal theory says that the price level is determined by the ratio of nominal debt to the present value of real primary surpluses. I analyze long-term debt and optimal policy in the fiscal theory. I find that the maturity structure of the debt matters. For example, it determines whether news of future deficits implies current inflation or future inflation. When long term debt is present, the government can trade current inflation for future inflation by debt operations; this tradeoff is not present if the government rolls over short term debt. I solve for optimal debt policies to minimize the variance of inflation. I find cases in which long-term debt helps to stabilize inflation, and I find that the optimal inflation-stabilizing policy produces time series that are surprisingly similar to U.S. surplus and debt time series.

John H. Cochrane
Graduate School of Business
University of Chicago
1101 E. 58th St.
Chicago, IL 60637
and NBER
john.cochrane@gsb.uchicago.edu

1 Introduction

The fiscal theory states that the price level is determined by the government budget constraint,

$$\frac{\text{nominal debt}}{\text{price level}} = \text{present value of real surpluses.} \tag{1}$$

The fiscal theory is developed by Leeper (1991), Sims (1994) Woodford (1995, 1996) and Dupor (1997) with one-period debt. Cochrane (1998) reviews the fiscal theory and argues for its plausibility.

At heart, the fiscal theory recognizes that even apparently unbacked fiat money is in the end valued by its backing from real government resources rather than by a liquidity value in exchange. In the fiscal theory, money and nominal debt are essentially valued as equity claims on the government. Well-understood backing regimes such as credible commodity standards and currency boards are transparent instances of the fiscal theory; the theory argues that their price determination mechanisms apply more broadly. As in these cases, fiscal price level determination is immune to private note issue and financial innovation in transactions technologies. This fact makes it an attractive positive and normative theory for the current situation—roughly stable prices and prodigious financial innovation.

In this paper, I extend the fiscal theory to include long-term debt. When long-term debt is present, the nominal value of the debt on the left hand side of (1) is not fixed; it depends on nominal bond prices which in turn depend on expected future price levels. To see why this fact might matter, suppose that there is bad news about future surpluses so the right hand side of (1) declines. If there is no long-term debt, the nominal value of government debt is fixed, so the price level must rise to re-equilibrate (1). However, if long term bonds are outstanding, their *relative* price and thus the numerator of the left hand side might fall instead. Lower bond prices correspond to expectations of higher future price levels, so long term debt can imply that bad news about future surpluses results in future rather than current inflation.

To analyze issues of this sort, I solve equations like (1) for the price level at each date, with surpluses and debt on the right hand side. I use the solutions to understand the obvious comparative statics exercises: 1) How does the price level react to current and future surpluses, holding debt constant? 2) How does the price level react to current and future debt holding surpluses constant? Answers to the first question are particularly useful in thinking about historical events such as currency crashes or the ends of hyperinflations. The second question starts us thinking about what alternative outcomes could have been in such situations.

In answer to the first question, I find that the effects of surpluses on the price level depend on debt *policy*: Current and expectations of future state-contingent debt sales and redemptions matter as well as the maturity structure of outstanding debt.

The effects are often surprisingly different than those in the short-term debt case. For example, when debt policy consists of paying off an outstanding perpetuity, the price level at each date is determined only by the surplus at that date. Past or future surpluses have no additional effect on the price level at all.

In answer to the second question, I find that the effects of debt on the price level also depend on the maturity structure and expectations of future debt policy. For example, suppose that the government sells some additional nominal debt. If no long-term debt is outstanding, the government faces a unit-elastic demand curve. Bonds are nominal claims to the same real resources, so bond prices fall one-for-one with the number sold; real revenue from bond sales and the price level today are unaffected by the number sold. However, if there are long-term bonds outstanding, selling extra debt dilutes the existing long term bonds as claims to the fixed stream of future real resources. Therefore, unexpected debt sales can raise revenue today and lower today's price level, even with no change in current or future surpluses.

Next, I consider what debt and surplus policies optimally smooth inflation, paying particular attention to motivations for long-term debt. The three elements of the government's policy choice are the average maturity structure, the choice of state-contingent debt sales and redemptions in response to fiscal shocks, and a limited control of the surplus. I add each element in turn and analyze the results in terms of the above comparative statics.

I start by analyzing optimal *passive* policy, in which the government determines only the steady state level of debt and its maturity structure, and the government does not adjust debt in response to surplus shocks. I find that short maturity structures are preferred when the present value of the surplus varies by less than the surplus itself; while long maturity structures are preferred when surpluses build up following a shock so that the present value varies by more than the surplus itself. This finding is a natural result of the comparative statics: the price level responds to the present value of surpluses with a short maturity structure, while the price level responds to the surplus at each date with a long maturity structure.

I then analyze optimal *active* policy, in which the government also changes the amount and maturity structure in response to surplus shocks. Now there is a second motivation for long-term debt. If long term debt is outstanding, the government can smooth inflation by occasionally and unexpectedly devaluing long-term bonds, trading a lower price level today for a higher price level in the future. This action can smooth inflation after a shock has hit. I study a quantitative example in which the optimal passive policy consists of short-term debt, but the optimal active policy includes long-term debt so that the government can smooth inflation by such ex-post devaluations.

I then add a limited control over the long-term surplus in order to better model the situation faced by the U.S. government. Actual policy almost always consists of simultaneous changes in debt and surpluses: Low surpluses are financed by extra nominal debt sales and extra nominal debt sales almost always come with implicit or explicit promises to increase future surpluses.

This optimal policy analysis solves some empirical puzzles. A simpleminded application of (1) and its comparative-static predictions for the effects of surplus and debt shocks seems disastrous for the fiscal theory in U.S. data. However, if we regard the U.S. government as solving such an optimal policy problem, *adapting* debt and fiscal policy to defend price level stability in the face of surplus shocks rather than *causing* price level disturbances by exogenous surplus and debt movements, we explain many of the initially puzzling features of the data.

For example, equation (1) suggests that the price level should move together with total debt. On the reasonable assumption that the present value of the surplus is higher when the surplus itself is high, it also suggests that the price level should move inversely with the surplus and that the real value of the debt should move together with the surplus. But none of these patterns is an even vaguely plausible description of U.S. data. Figure 1 presents the primary Federal surplus/consumption ratio and CPI inflation. If anything there is a slight positive correlation between surplus and inflation or price level growth at business cycle frequencies. (This data is presented in more detail in Cochrane 1998. Dividing by consumption gives a more plausibly stationary series, and the theory adapts easily to this transformation.) Figure 2 presents the surplus/consumption ratio together with the level and difference in total real value of the debt. Comparing the two figures, we can see that there is little correlation between the level of debt and the price level, inflation, or the surplus, as debt moves much more slowly than any of the other series. The surplus is nicely *negatively* correlated with changes in debt. Perhaps unsurprisingly, high surpluses pay down the debt.

By contrast, I find that the optimal policies produce time series that are similar to these U.S. time series in many dimensions. For example, the optimal policies generate a negative correlation between surpluses and debt growth, as in the data. The government smooths the effects of a negative surplus shock by issuing long-term debt, and by promising to raise future surpluses.

I close the paper by returning to solutions of equations like (1) for the price level given the sequence of surpluses and debt policy. All of the analysis described so far uses a convenient linear approximation. I derive a solution that is general and exact, though algebraically cumbersome, and I compare it to the approximate solution.

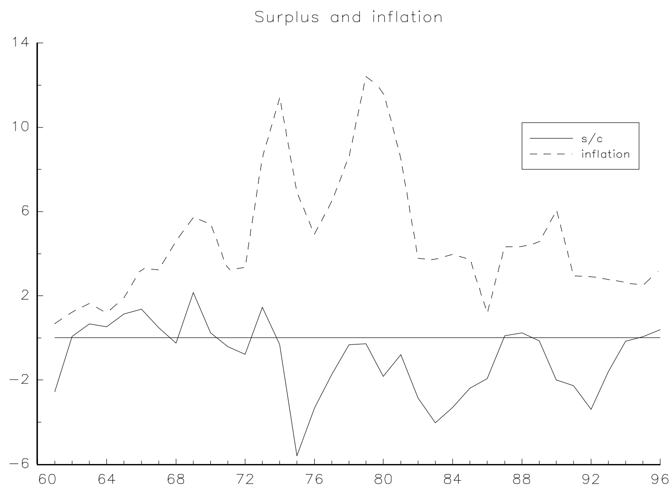

Figure 1: Federal primiary surplus / nondurable + services consumption and CPI inflation.

Figure 2: Surplus/consumption ratio, real value of the debt / consumption and difference of real value/consumption.

2 Definitions and identities

Let $B_t(j)$ denote the face value of zero-coupon nominal bonds outstanding at the end of period t that come due in period j. Let $Q_t(j)$ denote the nominal price at time

t of a bond that matures at time j. Of course, $Q_t(t) = 1$ and $B_t(j) = 0$ for $j \leq t$. Let p_t denote the price level and let s_t denote the real primary surplus, i.e. tax collections less government purchases.

I simplify the analysis by assuming that the expected real rate of return on government debt is equal to a constant r across time and maturity, and I denote the corresponding discount factor $\beta = 1/r$. Bond prices are therefore equal to

$$Q_t(t+j) = \beta^j E_t \left(\frac{p_t}{p_{t+j}} \right). \tag{2}$$

I also simplify the analysis by assuming a frictionless economy in which no cash is held overnight. The economy need not be "cashless," transactions may be facilitated by money created each morning and retired each night rather than by direct exchange of maturing bonds, and any amount of private money, bonds, banknotes, checking accounts etc. may be created with no effect on the government budget constraint and hence no effect on the price level. Since the models are frictionless, standard Modigliani-Miller theorems by which the maturity structure of the debt is irrelevant for *real* quantities still apply. Here I study the effects of the maturity structure on the *nominal* price level, and such effects can occur even in a frictionless economy.

The entire analysis flows from two equivalent identities. The *flow identity* says that the surplus must equal bond redemptions plus net repurchases,

$$\frac{B_{t-1}(t)}{p_t} - \sum_{j=1}^{\infty} \beta^j E_t \left(\frac{1}{p_{t+j}} \right) \left[B_t(t+j) - B_{t-1}(t+j) \right] = s_t, \tag{3}$$

while the *present value* identity says that the real value of outstanding debt equals the present value of real surpluses

$$\frac{B_{t-1}(t)}{p_t} + \sum_{j=1}^{\infty} \beta^j E_t \left(\frac{1}{p_{t+j}} \right) B_{t-1}(t+j) = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}. \tag{4}$$

The appendix presents a derivation of these equations. I use whichever form is more convenient for a given application. The content of equations (3) and (4) is an accounting identity plus the assumption of constant expected real returns. Lacking a better word, I call them "identities."

A *solution* is the sequence of prices $\{p_t\}$ that solves either (3) or (4) at each date and state, for a given sequence of debt policies $\{B_t(t+j)\}$ and surpluses $\{s_t\}$. In simple terms, a solution is an equation with p_t on the left and other quantities are on the right. Because prices multiply quantities in (3)-(4), such solutions are not trivial to find.

3 The effect of surpluses on the price level

In several special cases of the debt policy, we can find price-level solutions easily and directly. These cases also allow us to characterize the effect of surpluses on the price level, holding debt constant.

3.1 One period debt

Suppose that the government only issues one period debt, rolled over every period. This is the standard case analyzed in the fiscal theory, for example Woodford (1995). All terms $B_{t-1}(t+j)$ other than $B_{t-1}(t)$ are zero. Then, the present value identity, (4), specializes to a solution directly,

$$p_t = \frac{B_{t-1}(t)}{E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}.} \tag{5}$$

With one period debt, future deficits affect the price level today. The price level today responds only to the present value of surpluses.

While this case is familiar to fiscal-theory readers, it is *not* generally true that the present value identity is also a solution. In general, there are additional bonds on the left hand side of (4) and hence terms in p_{t+j}.

3.2 No new debt

Suppose instead that a full maturity structure is outstanding at time 0, and the government neither issues debt nor retires debt before it is due. For example, the government could pay off a perpetuity. In this case, debt due at t is constant over time, $B_{t-1}(t) = B_{t-2}(t) = B_0(t)$. The *flow* identity (3) is now also a solution,

$$p_t = \frac{B_{t-1}(t)}{s_t}. \tag{6}$$

Now, prices are determined by bonds that fall due at each date divided by that date's surplus. Shocks to future deficits have no influence at all on the current price level. Instead, long-term bond prices, reflecting future inflation, entirely absorb the shocks to the present value of surpluses. To see this fact, apply (6) at $t+j$; a shock to expected s_{t+j} changes expected $1/p_{t+j}$ and thus changes bond prices $Q_t(t+j) = \beta^j E_t\left(\frac{p_t}{p_{t+j}}\right)$.

3.3 k-period debt

As an intermediate example, suppose that each period the government issues $B_t(t+k)$ k-period discount bonds each period, and then lets them mature. With this debt policy, $B_t(t+k) = B_{t+1}(t+k) = \cdots = B_{t+k-1}(t+k)$. The flow identity (3) then becomes

$$\frac{B_{t-k}(t)}{p_t} - \beta^k E_t\left(\frac{1}{p_{t+k}}\right) B_t(t+k) = s_t.$$

This is a k-period difference equation, with solution

$$p_t = \frac{B_{t-k}(t)}{E_t \sum_{j=0}^{\infty} \beta^{jk} s_{t+jk}} = \frac{B_{t-1}(t)}{E_t \sum_{j=0}^{\infty} \beta^{jk} s_{t+jk}}.$$

The price level is still determined by a sort of present value, but only every kth term matters! For example, if the government issues 5 year debt, then expectations of surpluses in years 5, 10, 15, etc. matter to today's (0) price level, but surpluses in years 4, 6 etc. do not matter. As $k \to 1$ we recover the one period debt solution (5) in which all future deficits matter. As $k \to \infty$, we recover the case (6) in which only today's surplus matters to today's price level.

3.4 Geometric maturity structure

A geometric pattern gives a tractable and reasonably realistic way to analyze a rich maturity structure. Suppose that the amount of debt outstanding at the beginning of t (end of $t-1$) that will mature at $t+j$ declines at a rate ϕ^j:

$$B_{t-1}(t+j) = B_{t+j-1}(t+j)\phi^j. \tag{7}$$

Equivalently, the fraction of debt that matures at date t, sold at date $t-j$, is fixed across time,

$$\frac{B_t(t+j) - B_{t-1}(t+j)}{B_{t+j-1}(t+j)} = \phi^{j-1}(1-\phi);\ j \geq 1. \tag{8}$$

To derive a solution for this debt policy, plug (7) into the present value identity (4), and plug (8) into the flow identity (3). Adding the first and $\phi/(1-\phi)$ times the second equations and solving for p_t we obtain the solution,

$$p_t = \frac{B_{t-1}(t)}{s_t + (1-\phi)E_t \sum_{j=1}^{\infty} \beta^j s_{t+j}}. \tag{9}$$

This example also nests the one period debt case and the no-change-in-debt case as ϕ varies from 0 to 1. For $\phi = 0$, there is only one period debt, and price is determined by the present value of surpluses. For $\phi = 1$, price is determined only by the surplus at each date.

4 An approximate solution

To evaluate more complicated examples, and to study the effects on the price level of *changing* debt policy, we need to find price solutions in a more general setting. Section 8 below presents an exact general solution, but it is algebraically complex, it hides much of the intuition, and it does not allow us to use standard linear time-series techniques. Therefore, I start with a linearized solution about a steady state with a geometric maturity structure.

Denote steady states by

$$p_t = p,\ B_{t-1}(t+j) = B\phi^j,\ s_t = s. \tag{10}$$

Evaluating (4) at the steady state yields a restriction between steady state parameters,

$$\frac{ps}{B} = \frac{1-\beta}{1-\beta\phi}. \tag{11}$$

Denote by $\tilde{x}_t$ the proportional deviation of each variable x_t from the steady state,

$$\tilde{p}_t = \frac{p_t - p}{p};\ \tilde{s}_t = \frac{s_t - s}{s};\ \tilde{B}_{t-1}(t+j) = \frac{B_{t-1}(t+j) - \phi^j B}{B}. \tag{12a}$$

(It turns out to be more convenient to scale debt by B rather than $B\phi^j$.) Differentiating the present value identity (4) about the steady state and using (11) then gives the approximate present value identity.

$$\sum_{j=0}^{\infty} \beta^j \phi^j E_t \tilde{p}_{t+j} = -\frac{1-\beta}{1-\beta\phi} \sum_{j=0}^{\infty} \beta^j E_t \tilde{s}_{t+j} + \sum_{j=0}^{\infty} \beta^j \tilde{B}_{t-1}(t+j). \tag{13}$$

Comparing this equation with its exact counterpart (4), we see that the approximation uses the steady state price level to value outstanding debt rather than the actual price levels, hence terms $\beta^j \tilde{B}_{t-1}(t+j)$ appear in place of $\beta^j E_t\left(1/p_{t+j}\right) B_{t-1}(t+j)$. It also uses the steady state maturity structure $B\phi^j$ rather than the actual maturity structure $B_{t-1}(t+j)$ to capture the trade-off between current and future price levels. As usual, linearizing a product gives the steady state of each term times the deviation of the other and ignores terms in which deviations are multiplied by each other.

Since the weights ϕ^j are geometric, iterating (13) forward to solve for $\tilde{p}_t$ is easy, and gives the approximate price solution,

$$\tilde{p}_t = -\left(\frac{1-\beta}{1-\beta\phi}\right)\left(\phi \tilde{s}_t + (1-\phi)\sum_{j=0}^{\infty} \beta^j E_t \tilde{s}_{t+j}\right) + B_{t-1} - \beta\phi B_t \tag{14}$$

where

$$B_{t-1} \equiv \sum_{j=0}^{\infty} \beta^j \tilde{B}_{t-1}(t+j).$$

The appendix gives some more details of this iteration.

In (14), we see again how the geometric maturity structure nests short and long term debt cases. If $\phi = 0$, price is proportional to the present value of the surplus; if $\phi = 1$, price is proportional to the surplus at each date.

The geometric *steady state* is distinct from the geometric *maturity structure* studied above. In the previous case, the maturity structure is always exactly geometric. In this case, debt of various maturities can wander away from the geometric steady state, and we can evaluate the effect on the price level of such wandering.

5 The effects of debt policy

This approximate solution (14) allows us to answer, what are the effects of *debt* changes, holding *surpluses* constant?

The first debt term in (14) means that an increase in debt at date $t-1$, B_{t-1}, that is repurchased at t (so that B_t does not also change) moves the price level p_t one for one. With one-period debt this effect is simple: more debt as a claim to the same fixed resources must result in a higher price level. The solution shows that more long-term debt at time $t-1$ also raises the price level at time t, even though the debt does not come due until later. Working through the definitions of B_{t-1} and $\tilde{B}_{t-1}(t+j)$, if maturity j debt $B_{t-1}(t+j)$ increases 1% relative to total steady state debt $\sum_k \beta^k \phi^k B$, the price level rises by $\beta^j/(1-\beta\phi)$percentage points. Thus, the effect of debt on the price level is attenuated for longer term debt and as the maturity structure shortens.

The second debt term in (14) means that an increase in debt at date t, B_t, can decrease the price level at time t, but *only* if some long-term debt is outstanding, i.e. if $\phi > 0$. If the government just rolls over short-term debt, this effect does not exist. New long-term debt dilutes outstanding long-term debt as a claim to fixed resources. The more long-term debt is currently outstanding, the less the dilution, and hence the more revenue the government can raise for each dollar of extra long-term bond sales. In turn, the more real revenue raised, the greater the impact on the price level.

In most cases the government does not sell long-term debt and then repurchase it one period later. Rather it sells additional long term debt and then lets it mature. To calculate the effects of such a policy, suppose that at time 0 the government sells an additional 10 year bond, starting from a geometric steady state maturity structure, and then lets that bond mature. This means $\tilde{B}_0(10) = \tilde{B}_1(10) = \tilde{B}_2(10) = ... =$

$\tilde{B}_9(10) = 1$. Using (14), the resulting price path is

$$\begin{aligned} \tilde{p}_0 &= -\beta^{10}\phi \\ \tilde{p}_t &= \beta^{10-t}(1-\phi)\,;\ t = 1, 2, ...9 \\ \tilde{p}_{10} &= 1. \end{aligned}$$

Figure 3 plots this price path. At date 0, we only have the second, negative debt term in (14); the price level is reduced if there is long term debt outstanding. At time 10, we only have the first, positive term in (14), so the price level rises by 1.0 for any maturity structure. One more bond must be redeemed from the same set of resources. In the intermediate dates, both terms in (14) are present. With long term debt, they cancel so there is no intermediate effect on the price level. With shorter term debt, the price level increases all the way out to period 10.

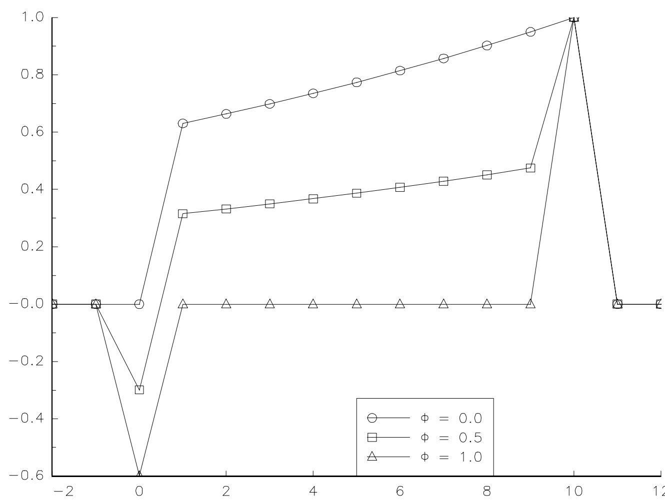

Figure 3: Effect on the price level of an increase in 10 period debt at time 0 that is allowed to mature, starting in a steady state with a geometric maturity structure.

5.1 Postponing inflation

As we have seen, additional sales of long term debt can lower the price level today while raising it in the future, when some long term debt is outstanding, even with no change in surpluses.

To what extent can the government affect the price level today through unexpected bond sales? For example, can it completely offset surplus shocks? The present value identity (4) answers these questions directly and exactly Rewriting the identity slightly,

$$\sum_{j=0}^{\infty} \beta^j E_t \left(\frac{1}{p_{t+j}} \right) B_{t-1}(t+j) = \sum_{j=0}^{\infty} \beta^j E_t(s_{t+j}). \tag{15}$$

We can read this equation as "budget constraint" for achievable expected inverse price levels. *The maturity structure of outstanding debt $B_{t-1}(t+j)$ gives the rates at which the government can trade off the price level today for price levels in the future.*

The government can always raise future prices by selling more debt; the issue is whether such sales affect today's prices. With outstanding long-maturity debt, terms $B_{t-1}(t+j),\ j \geq 1$ in (15) are present, so that raising future price levels (by selling more long term debt) can lower today's price level. If only one period bonds are outstanding, these terms are absent so there is nothing the government can do with debt policy to affect prices today.

Furthermore, *there is a debt policy – a choice of* $\{B_t(t+i),\ B_{t+1}(t+i)...;\ i = 1, 2, ...\infty\}$ *that achieves any set of price paths consistent with the constraint* (15). To verify this fact, we can construct a policy that works for a given price path. It is not unique. Let the government adjust its maturity structure once, determining $B_t(t+j)$ and then making no further changes. Future price levels are given by the solution (6), and taking expectations at time t,

$$E_t \left(\frac{s_{t+j}}{B_{t+j-1}(t+j)} \right) = E_t \left(\frac{1}{p_{t+j}} \right).$$

Therefore, if the government sets

$$B_t(t+j) = \frac{E_t(s_{t+j})}{E_t \left(\frac{1}{p_{t+j}} \right)}$$

the desired path of future price levels $\{E_t (1/p_{t+j})\}$ results. Then (15) produces the price level at time t.

The converse statement is also true. *If there is no j period debt outstanding at time t, then there is no debt policy – no choice of $\{B_{t+1}(t+i),\ B_{t+2}(t+i)...; i = 1, 2, ...\infty\}$ — by which the government can lower the price level at time t in exchange for raising the price level at time $t+j$.*

Can the government go so far as to attain a *constant* price level in the face of surplus shocks by appropriately buying and selling bonds? The constraint (15) shows that this much is not possible, because debt at time t must be in the time t information set. Take innovations of equation (15), resulting in

$$\sum_{j=0}^{\infty} \beta^j (E_t - E_{t-1})(s_{t+j}) = \sum_{j=0}^{\infty} \beta^j B_{t-1}(t+j)(E_t - E_{t-1}) \left(\frac{1}{p_{t+j}} \right)$$

A constant price level implies $(E_t - E_{t-1})\left(\frac{1}{p_{t+j}}\right) = 0$ for all j. The right side is zero and the left side is not, so this cannot be a solution. This conclusion holds in continuous time versions of the model as well.

5.2 Complete markets and commodity standards

With state-contingent debt, the government can attain a constant price level via debt policy alone despite surplus shocks. For example, suppose that the government has issued state-contingent debt at time 0 and engages in no further debt sales or repurchases. Let $B(\sigma^t)$ denote the amount (positive or negative) of nominal debt that comes due at date t in state σ^t. Similarly, let $s(\sigma^t)$ denote the real surplus at time t in state σ^t. The budget identity at each date is then simply

$$p(\sigma^t)s(\sigma^t) = B(\sigma^t).$$

In this case, the government can attain any stochastic process for prices, including a constant price level, by choosing the appropriate state-contingent debt structure.

A constant price level is not possible with non-state-contingent debt, even when we allow dynamic trading and long-term debt. Therefore, though dynamic trading of long term debt allows a greater array of state-contingencies than short term debt, it does not attain the complete-markets limit. In this paper, I focus on non-state-contingent nominal debt because that is the nearly universal structure of nominal government debt.

Similarly, if the government financed its deficits with indexed debt, if explicit default rather than inflation were the habitual mechanism for adjusting the value of the debt to the value of surpluses, or if government debt was not used as numeraire, then the fiscal theory would have little to say about the price level. It is relevant because governments issue (almost) default-free nominal debt.

Commodity standards are an instance of the fiscal theory of the price level, and they can give a constant price level as well. How is this consistent with the above statement? Commodity standards do endogenize government debt, since people can trade goods for debt freely, but this is not how they determine a constant price level. Commodity standards are also a commitment device for *surpluses*. If there is a transitory adverse surplus shock, in order to maintain the commodity standard forever the government *must* change policy so that the present value of future surpluses is unchanged.

6 Optimal debt policy

We have seen that debt policy can affect current and future inflation. Now I examine *optimal* policies that smooth inflation. I proceed in three stages: First, I find an optimal *passive* debt policy, i.e. an optimal steady state maturity structure, given that the government does not adjust debt ex-post in response to shocks. Then, I allow the government to additionally pursue *active* debt policy, adjusting the level of debt of various maturities in order to offset surplus shocks. Finally, I allow the government to control part of the surplus as well.

We can anticipate the qualitative results. As we have seen, shorter maturity structures relate today's price to many leads of the surplus; such a maturity structure smooths inflation if surpluses have a transitory component. Long maturity structures relate today's price to fewer leads of the surplus. Therefore, a passive policy with a long maturity structure smooths inflation better than a passive policy with a short maturity structure when surpluses build following a shock so that the present value is more volatile than the actual value, and vice versa. Long maturity structures also make active debt policy possible, so that the government can smooth a surplus shock as it happens by selling more long-term debt. This fact weighs in favor of a long maturity structure, even when short term debt is the optimal passive policy.

6.1 Statement of the problem

Given a stochastic process for the surplus $\{s_t\}$, the government picks the parameters governing the steady state maturity structure ϕ, B and a debt policy $\left\{\tilde{B}_t(t+j)\right\}$ to minimize the variance of inflation,

$$min\left[var\left(\tilde{p}_t - \tilde{p}_{t-1}\right)\right] \tag{16}$$

given that prices are generated by the solution (14). The steady state maturity parameter ϕ must respect the restriction $0 < \phi < 1$ and the debt choice $\tilde{B}_t(t+k)$ must be in the time t information set, and must be a stationary (not explosive) process. (The notation is defined in (10)-(12a).)

I state the objective and constraints in terms of steady states and deviations about the steady state, since I use the approximate price solution to solve the problems. In order to use the approximate solution, I constrain the government's choice to a geometric steady state.

Smoothing the volatility of inflation is a reasonable characterization of central bank objectives. In this model, the level of inflation is arbitrary and so it is not interesting to add it to the objective. I follow a long tradition in monetary economics, for example Sargent and Wallace (1975), and do not delay or complicate the analysis by justifying the inflation-smoothing objective from welfare maximization in an economy with specific frictions. Modeling "inflation" as the difference of proportional deviations from the steady state as in (16) rather than the ratio of price levels is an analytically convenient simplification.

The methods adapt easily to other objectives. For example, one can minimize the variance of the price level, which may describe prewar or gold-standard policy. Alternatively, one can minimize the variance of unexpected inflation $\min var(\tilde{p}_t - E_{t-1}\tilde{p}_t)$, motivated by the Lucas (1972, 1973) world in which only unexpected money has real effects.

6.2 Passive policy

I start by analyzing *passive* policies: The government chooses only a steady state maturity structure, governed by the parameters B, ϕ, in order to minimize the variance of inflation given that prices are generated by (14). I calculate results for an AR(2) surplus process,

$$\tilde{s}_t = (\lambda_1 + \lambda_2)\tilde{s}_{t-1} - (\lambda_1\lambda_2)\tilde{s}_{t-2} + \varepsilon_t.$$

Figure 4 presents the optimal steady state maturity parameter ϕ as a function of the two roots λ_1 and λ_2.The calculation is detailed in the appendix. The overall level of debt B simply governs the steady state price level p, and so is irrelevant to the inflation-smoothing objective.

For every stationary AR(1) (one root equal to zero, the other strictly less than one) the optimal maturity is short, $\phi = 0$. In these cases the variance of the present value of the surplus is smaller than the variance of the surplus, so short term debt smooths inflation by making the price level equal to the smoother series. Two large roots λ produce hump-shaped impulse response functions that continue to rise after an initial shock, and for which the present value varies by more than the series itself. In this region, the longest possible maturity debt $\phi = 1$ is preferred, because long term debt makes the price level proportional to the surplus at each date.

Most interestingly, there is a region with two reasonably large roots λ_1 and λ_2 for which a maturity structure with ϕ intermediate between 0 and 1 is optimal. This case is not implausible, as many macroeconomic time series have hump-shaped impulse-response patterns with roots roughly those of this region.

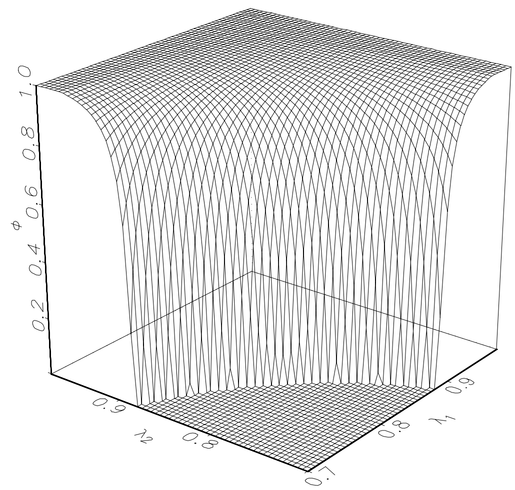

Figure 4: Optimal geometric maturity ϕ of passive debt policies that minimizes the variance of inflation, as a function of the two roots of the AR(2) surplus process. $\beta = 0.95$.

6.3 Active policy

To study active policies, I further specialize to an AR(1) surplus process

$$\tilde{s}_t = \rho \tilde{s}_{t-1} + \varepsilon_t$$

The methods generalize to arbitrary processes but this case will provide enough interesting behavior, and more than enough algebraic complexity. The problem is now to choose $\phi \in [0,1], \{B_t\}$ to minimize $var(\tilde{p}_t - \tilde{p}_{t-1})$, given that, using the AR(1) surplus in (14), $\tilde{p}_t$ is generated by

$$\tilde{p}_t = -k(\phi)\tilde{s}_t + B_{t-1} - \phi\beta B_t \tag{17}$$

where

$$k(\phi) \equiv \frac{(1-\beta)(1-\beta\phi\rho)}{(1-\beta\rho)(1-\beta\phi)}.$$

The solution to this problem, derived in the appendix, is the following policy:

$$\begin{aligned} \phi &= 1 \\ (1-L)(1-\beta L)B_t &= (-\gamma + \beta L)\, s_t, \end{aligned} \tag{18}$$

where

$$\gamma \equiv \frac{1-\rho+\beta(1-\beta)}{1-\rho\beta}.$$

Note from (18) that debt B_t depends on the whole history of s, despite the AR(1) structure.

Note that a long maturity structure $\phi = 1$ is in fact optimal in this case, even though $\phi = 0$ or short term debt is the optimal passive policy for an AR(1) surplus. Long term debt makes active debt policy possible, and the ability to offset shocks as they come by diluting and devaluing outstanding long-term debt dominates the passive inflation-smoothing properties of a short maturity structure.

The active policy fundamentally transforms the price level process. While the price level would follow the surplus AR(1) with a passive policy, now *inflation* follows an AR(1),

$$(1-L)\tilde{p}_t = -\frac{(1-\beta^2)}{(1-\rho\beta)}\frac{(1-\beta)}{(1-\beta L)}\varepsilon_t$$

By making the price *level* nonstationary, *inflation* can be smoothed.

If the government minimizes the variance of the price level rather than that of inflation, optimal debt policy produces artificial time series reminiscent of prewar or gold-standard time series. In place of a unit root price level and smooth inflation, this objective produces a price level with low autocorrelation, and inflation that varies a great deal. Thus, the shift in the character of inflation in the U.S. between the prewar and postwar period can be understood as a shift from a price-level targeting objective to an inflation smoothing objective.

Figure 5: Artificial data from optimal active debt policy with an AR(1) surplus.

Figure 5 presents artificial time series for debt growth, surplus and price level for this model. Debt has a unit root, while inflation and the surplus are stationary. As in the data, there is no visible correlation between debt or the surplus and the price level. As in the data, nominal debt growth is negatively correlated with the surplus.

Figure 6: Artificial inflation data. "Active" gives inflation from the optimal active debt policy with an AR(1) surplus. "Passive, long term debt" gives inflation with the long ($\phi = 1$) maturity structure of the optimal active policy, but uses only a passive policy; debt is always equal to the steady state value. "Passive, short term debt" gives inflation with short term debt ($\phi = 0$), which is the optimal passive policy for an AR(1) surplus.

Figure 6 contrasts inflation from the optimal active policy with the inflation that would result from a passive policy with long term debt ($\phi = 1$) and from the optimal passive policy, which uses short term debt ($\phi = 0$). With either passive policy, the price level is perfectly correlated with the surplus, and so inflation is perfectly correlated with surplus growth. Comparing active and passive policy with long term debt, we see that active policy dramatically smooths inflation. Active policy also produces inflation smoother than the optimal passive policy, which uses short-term debt.

Following a negative surplus shock, the government sells additional long term debt. This action lowers the price level at the moment of the shock, but raises the price level in the future. The result is a smoother path of *inflation* at the cost of a more volatile – a unit root in fact – price *level*.

However, the surplus is still positively correlated with the *real* value of the debt in this model, as it must in any AR(1) surplus model. To match the fact in the data that both real and nominal debt growth are negatively correlated with the surplus, I consider surplus policy below.

7 Optimal surplus and debt policy

So far, I have examined the price level effects of changing surpluses with constant or exogenous debt, and then I have examined the effects of changing debt with constant or exogenous surpluses. These are natural places to start analyzing the logical possibilities implied by the fiscal theory, but neither is a realistic descriptions of actual debt policy. Governments have at least some control over the surplus as well as nominal debt, and a realistic policy optimization exercise should recognize this fact. Most importantly, the vast majority of debt sales come together with an implicit or explicit promise to increase future surpluses.

A second and related issue is that the AR(1) or AR(2) surplus processes investigated above, though they are natural examples and plausible descriptions of the *univariate* behavior of the U.S. real primary surplus, lead to a completely counterfactual description of the *joint* behavior of surplus and debt. Simple AR surplus processes imply that the value of the debt should be positively correlated with surpluses. Higher current surpluses mean a higher present value of future surpluses and hence a higher value of the debt. In the data, as shown in Figure 2, high surpluses are unsurprisingly associated with *declining* real debt. Cumby Canzoneri and Diba (1998) use these counterfactual predictions of AR(1) surplus processes to reject the fiscal theory.

To make this point precisely, denote the real value of the debt v_t. The present value identity (4) says that the real value of the debt – of any maturity structure – is equal to the present value of real surpluses.

$$v_t = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}. \tag{19}$$

With an AR(1) surplus, $s_t = \rho s_{t-1} + \varepsilon_t$, the surplus and real value of debt are perfectly correlated.

$$v_t = \frac{1}{1-\beta\rho} s_t \tag{20}$$

This result is true for *any* debt policy, including the active debt policy analyzed above. More generally, any time series process in which the present value on the right hand side of (19) moves positively with the series itself predicts that surpluses should be positively correlated with debt.

Therefore, to plausibly describe the joint behavior of surplus and real debt, the surplus *must* follow a process whose level is negatively correlated with long-run and present values. This statement has nothing to do with the fiscal theory, since equation (19) is entirely in real terms and holds in all models, fiscal or not.

On first glance processes with negative long-run responses seem strange. On second glance they suggest that surpluses respond to real debt values, the "Ricardian regime" special case that invalidates the fiscal theory. But on third glance such processes are a natural outcome of a debt policy run to smooth inflation in the face of transitory surplus shocks.

In a recession, the government must finance a deficit. It can do one of three things:

1. It can inflate away existing debt. For example, with one-period debt we have
$$\frac{B_{t-1}(t)}{p_t} + E_t\left(\frac{1}{p_{t+1}}\right)\beta B_t(t+1) = s_t.$$
If the government does not change nominal debt $B_t(t+1)$ and future surpluses s_{t+j}, a negative s_t shock will be met by a rise in p_t, i.e. by inflating away the real value of outstanding debt.

2. As discussed above, if long-term debt is outstanding, the government can sell additional long term debt with no change in future surpluses; this action devalues outstanding long term debt, causing future rather than current inflation.

3. The government can sell additional debt, *while promising to increase future surpluses.* For example, with one-period debt, an increase in debt sales $B_t(t+1)$ while holding future surpluses s_{t+1} constant results in an equiproportionate increase in the future price level p_{t+1} and hence does not raise any revenue or affect prices at time t. But if the government can promise to raise future surpluses, then it can sell more debt $B_t(t+1)$ with no effect on p_{t+1}; hence it can raise more revenue without inflating away existing debt. In this last example a negative surplus shock today is followed by an increased surplus in the future.

The first two options lead to large swings in inflation. The third strategy leads to much less volatile inflation. Hence, we expect a government that wishes to smooth inflation to follow something like the third strategy. And in fact we routinely think of governments offsetting current fiscal stringency by borrowing, and implicitly or explicitly promising to raise future taxes or cut future spending to pay off the resulting debt. Thus, we routinely think of surplus processes, which, under partial government control, have response functions which reverse sign after a shock.

The first two options also lead to real values of the debt that are positively correlated with the surplus. The fact that high surpluses seem to pay down the real value of the debt is not an accounting identity; it results from the government's choice to do so rather than to meet surpluses with inflation.

7.1 A model of optimal fiscal policy

Here, I pursue a model that captures the intuition of the last few paragraphs. First, we must describe the surplus process. There is a cyclical component to the surplus that is by and large beyond the government's control. In a recession, lower income at constant tax rates means less tax revenue, and entitlement and other program-based spending automatically rise. Denote this cyclical portion of the surplus

$$c_t = \rho c_{t-1} + \varepsilon_t \tag{21}$$

The government does control a long-term component of the surplus. By changing tax rates and the terms of government programs, it alters the overall level of the surplus. For good optimal-taxation reasons it does not change tax rates and spending policies to offset the transitory, cyclically-induced component of the surplus, for example raising tax rates in recessions and lowering them in booms. Let the controllable component of the surplus follow a random walk,

$$z_t = z_{t-1} + \delta_t. \tag{22}$$

The actual surplus is the sum of the two components,

$$\tilde{s}_t = c_t + z_t.$$

(The random walk is merely a convenient simplification. The model works in much the same way if z_t follows any process $z_t = \eta z_{t-1} + \delta_t$ that is more persistent than c_t, $\eta \gg \rho$ so that z_t controls the long-run surplus.)

Next, we must state the government's problem. The government now picks the change in the controllable component of the surplus δ_t at each date. δ_t must be in the time-t information set, and it must not be predictable from time $t-1$ information. The government also picks nominal debt B_t in the time t information set, stationary (not explosive) as above, and the steady state maturity structure ϕ. (Once again the steady state level of debt B affects only the steady state price level.) The government picks $\phi, \{\delta_t\}, \{B_t\}$ to minimize the variance of inflation, given that the price level is determined by (14), which specializes given this surplus structure to

$$(1-L)\tilde{p}_t = -\frac{(1-\beta)(1-\beta\phi\rho)}{(1-\beta\rho)(1-\beta\phi)}(1-L)c_t - \delta_t + (1-L)(L-\phi\beta)B_t. \tag{23}$$

Now we can study solutions to this problem. There are policies that set the variance of inflation to zero. The government may choose ϕ arbitrarily (the optimum policy is not unique) and then chooses debt and the long-run component of the surplus according to

$$(1-L)B_t = -\frac{1-\beta}{1-\beta\phi}\frac{1-\rho}{1-\beta\rho}c_t \tag{24}$$

$$\delta_t = -\frac{1-\beta}{1-\beta\rho}\varepsilon_t. \tag{25}$$

To check this solution, plug these choices into (23) and verify that each power of L on the right hand side is equal to zero.

7.2 Character of the solution

From (25), shocks to the long run surplus are negatively correlated with shocks to the transitory component of the surplus. As expected, the government meets a short run negative surplus shock by raising surpluses in the long run.

7.2.1 A graph of artificial data

Figures 7 and 8 plot simulated time series from the optimal policy system. The parameters are $\rho = 0.6$ and $\beta = 0.95$. The pictures are identical for any value of $\phi \in [0, 1]$. The random number draw is the same across the two pictures.

In Figure 7 we see how the surplus is generated from its permanent and transitory components. There are periodic recessions, in which the transitory component of the surplus declines, and booms in which it rises. The government slightly raises the permanent component of the surplus in the recessions and lowers it in the booms. This change has little effect on the short run properties of the surplus, since the actual surplus tracks the transitory component closely. But it has a dramatic effect on the long-run or present value properties of the surplus. The long-run surplus *rises* in recessions so the government can raise revenue by selling debt, and it *falls* in booms as the government pays off debt.

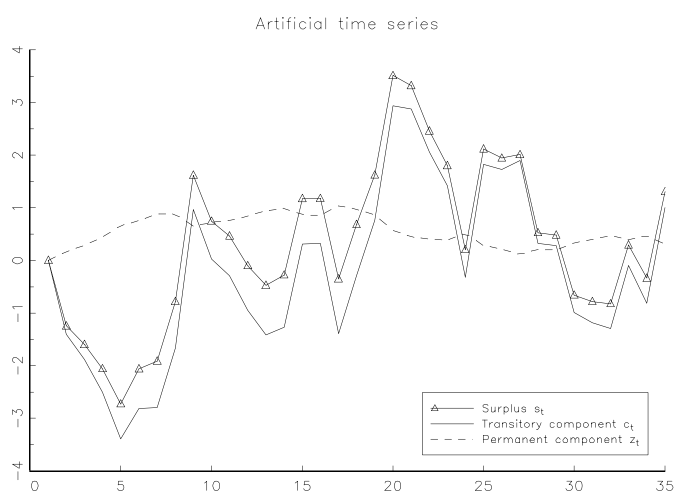

Figure 7: Simulated surplus, and its permanent and transitory components.

Figure 8 presents the joint properties of the total surplus, debt and debt growth. Comparing Figure 8 to actual data in Figure 2 we notice the striking similarity. Debt is not well correlated with the surplus, and it wanders at much lower frequency than the surplus; *growth* in debt is nicely negatively correlated with the surplus. The simple model thus accounts for the initially puzzling time-series behavior of debt and surplus, and shows why despite a simple AR(1) input, the result is far from the perfect positive correlation of debt and surplus that a pure AR(1) surplus process predicts. (Since the quantities B, s denote are proportional deviations from steady state, Figure 8 presents

$$\hat{B}_t \equiv \frac{B}{ps} B_t = \frac{1-\beta\phi}{1-\beta} B_t.$$

This transformation converts the debt series to the same units – real and relative to the surplus steady state – as the surplus series. This transformation also completely removes ϕ from the time-series properties of $\hat{B}_t, \tilde{s}_t$.)

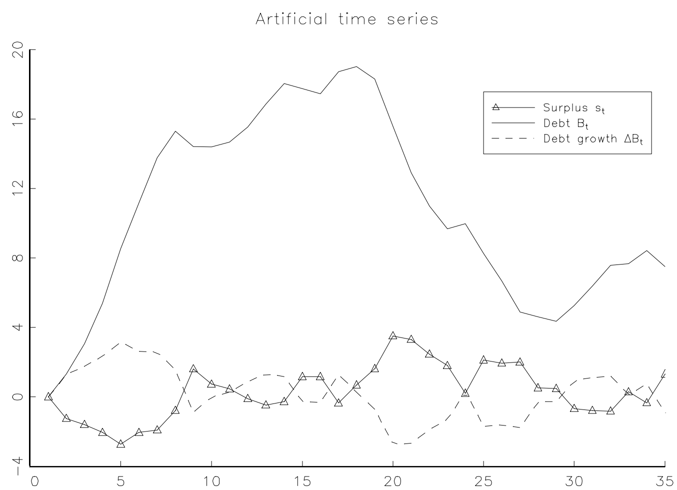

Figure 8: Simulated surplus, debt and debt growth.

7.2.2 Time series processes

For a slightly more formal comparison of model and data, we can compare the time-series process of debt and surplus predicted by the simple model to those we can estimate in the data. Then debt and surplus in the model follow the joint time series process

$$(1-\rho L)(1-L)\hat{B}_t = -\frac{(1-\rho)}{(1-\beta\rho)}\varepsilon_t \tag{26}$$

$$(1-\rho L)(1-L)\tilde{s}_t = \frac{(1-\rho)}{(1-\beta\rho)}(\beta - L)\varepsilon_t \tag{27}$$

and hence the two series are related by

$$\tilde{s}_t = -(\beta - L)\hat{B}_t. \tag{28}$$

These relations hold for any value of the steady state maturity structure ϕ.

Of course, inflation is not constant in actual data, and there is no linear function linking debt and surplus with no error term. Therefore, a formal test of (26)-(28) rejects the model. Nonetheless, we can see to what extent this model captures features of the data, as the above graphs suggest it does.

Debt process Table 1 presents regression estimates of the total debt process (26). The Table verifies that an AR(2) with one root near unity and one root around 0.5 is an excellent fit to this process.

	$\hat{B}_{t-1}$	$\hat{B}_{t-2}$	$\hat{B}_{t-3}$	R^2	DW
$\hat{B}_t =$	1.42	-0.49		0.93	2.16
t-stat.	(9.1)	(-3.2)			
$\hat{B}_t =$	1.29	-0.14	-0.23	0.93	1.94
t-stat.	(6.99)	(-0.43)	(-1.25)		

	$\Delta\hat{B}_{t-1}$	$\Delta\hat{B}_{t-2}$	R^2	DW
$\Delta\hat{B}_t =$	0.46		0.18	2.06
t-stat.	(2.90)			
$\Delta\hat{B}_t =$	0.36	0.18	0.18	1.91
t-stat.	(1.94)	(0.96)		

Table 1. Autoregressions of total debt/consumption ratio, 1960-1996.

Debt-surplus relation Equation (28) is consistent with the finding in the data that the surplus is strongly negatively correlated with *changes* in the total value of the debt, and given the debt process (26), poorly correlated with the level of the total value of the debt. To quantify this relation, Table 2 presents a regression of surplus on debt.

	$\hat{B}_t$	$\hat{B}_{t-1}$	R^2	DW
$s_t =$	-0.44	0.48	0.61	2.15
t-stat.	(-6.65)	(7.34)		

Table 2. Regression of surplus/consumption ratio on total debt 1960-1996.

The relative values of the coefficients on current and lagged debt conform to the prediction of (28). The absolute values are about a half too small. There is of course no error in (28), while there is an error in the actual data. The data for Table 2 obey the identity

$$s_t = r_t \hat{B}_{t-1} - \hat{B}_t$$

where r_t is the gross ex-post real return on the government bond portfolio less the consumption growth rate. Therefore, the error term in the regression is largely the real return on government bonds. That return was low in the first half of the sample, when the surplus and right hand side of (28) was high, and high in the latter part of the sample when the surplus and right hand side of (28) was low. There is a decade-long movement in the error term, correlated with the right hand variable. This fact lowers both coefficients but does not affect their relative values.

Surplus process The surplus/consumption ratio is well- modeled as an AR(1), or at most an AR(2). Table 3 presents autoregressions. The autocorrelation function also has a classic AR(1) shape, with at most a small secondary hump with t-statistics around 1.5.

	s_{t-1}	s_{t-2}	DW
$s_t =$	0.56		
t-stat.	(3.93)		1.62
$s_t =$	0.72	-0.23	
t-stat.	(4.22)	(-1.35)	1.99

Table 3. Autoregressions of the surplus/consumption ratio

Equation (27) represents the evolution of s_t from shocks to the bivariate $\{s_t, B_t\}$ system. However, since $\beta < 1$ the moving average term is not invertible. Hence, this is not the univariate Wold representation as would be recovered by autoregressions or univariate ARMA estimation. The univariate Wold representation is[1]

$$(1-L)\tilde{s}_t = \left(\frac{1-\beta L}{1-\rho L}\right)\eta_t;\ \eta_t = \tilde{s}_t - Proj(\tilde{s}_t|\tilde{s}_{t-1}, \tilde{s}_{t-2}...). \tag{29}$$

Figure 9 plots the univariate (response to η) and multivariate (response to ε) response functions. The univariate response function is very close to an AR(1):

[1]To find the univariate representation, write the spectral density of (27)

$$\begin{aligned} S_{(1-L)\tilde{s}_t}(z) &= \left(\frac{1-\rho}{1-\beta\rho}\right)^2 \beta^2 \frac{(1-\frac{1}{\beta}z)(1-\frac{1}{\beta}z^{-1})}{(1-\rho z)\,(1-\rho z^{-1})}\sigma_\varepsilon^2 \\ &= \left(\frac{1-\rho}{1-\beta\rho}\right)^2 \frac{(1-\beta z^{-1})(1-\beta z)}{(1-\rho z)\,(1-\rho z^{-1})}\sigma_\varepsilon^2. \end{aligned}$$

The roots of the last expression are all stationary, so this corresponds to the Wold representation.

$\beta \approx 0.95$ so the unit root on the left hand side nearly cancels the moving average root on the right hand side, leaving only the autoregressive root $(1-\rho L)$. At long horizons, the univariate response function stops decaying at a positive value $(1-\beta)/(1-\rho) = 0.125$ so it is in fact even more persistent than an AR(1). A researcher examining the univariate properties of s_t from this model would undoubtedly stop at an AR(1); most diagnostics are not capable of noticing the long-run divergence from an AR(1) implied by the near-canceling of roots. Thus, the univariate surplus process is broadly consistent with the data.

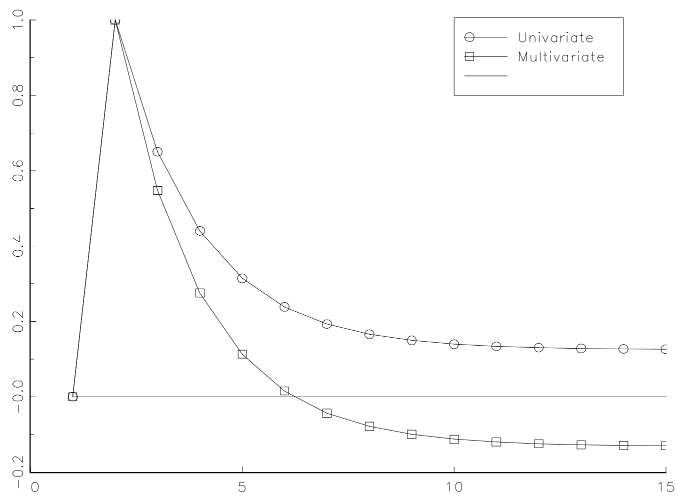

Figure 9: Response of the surplus to univariate Wold representation shocks η_t and to fundamental, multivariate shocks ε_t. Parameters are $\beta = 0.95$, $\rho = 0.6$.

A subtle trap for empiricists Figure 9 reminds us of a very subtle trap for empiricists. What could be more natural in evaluating the fiscal theory than to fit a surplus process, take its expected present value, and then test whether the real value of the debt does indeed correspond to the estimated present value of the surplus? A reader of Hansen, Roberds and Sargent (1991) already knows that one cannot follow this procedure; present values in such a test must be calculated from the *joint* debt-surplus process, because the univariate surplus model cannot reveal agent's information sets. Furthermore, we have already seen in (29) that the shock to agents' information sets cannot be recovered from current and past surpluses. Figure 9 shows what will go wrong if we try to take present values using the univariate process: The univariate response is always positive, while the true response function to shocks to agents' information is eventually negative. Thus "present values" calculated from responses to the univariate shock move positively with the surplus itself, while the true present value moves negatively with surpluses.

To give a better feel for this problem, Figure 10 plots simulated surplus time series along with the true value of the debt, the value predicted by an AR(1) and the value predicted from the correct univariate process. The true value of the debt is equal to the true present value of the surplus, $v_t = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}$ as in previous plots. The AR(1) debt prediction uses the AR(1) model $s_t = \rho s_{t-1} + \varepsilon_t$ to calculate the present value

$$v_t^{AR(1)} = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j} = s_t/(1-\beta\rho).$$

As the graph shows, this calculation predicts a value of the debt that is perfectly correlated with the surplus, and nothing at all like the true value of the debt. The univariate debt prediction uses the true univariate surplus process (29) rather than the AR(1) approximation to calculate the present value of the surplus,[2]

$$v_t^{\text{univariate}} \equiv E\left(\sum_{j=0}^{\infty} \beta^j s_{t+j} \middle| s_t, s_{t-1}, s_{t-2}, \ldots\right) = \frac{(1+\beta)}{(1-\beta\rho)} \frac{\left(1-\beta\frac{1+\rho}{1+\beta}L\right)}{(1-\beta L)} s_t \tag{31}$$

This prediction for the value of the debt is again positively correlated with the surplus and has no resemblance to the true debt process.

In sum, a researcher who fit a univariate surplus model and compared its present value to the value of the debt, using data from this artificial economy, would reject the present value identity. He would most likely fit an AR(1), coming to the dramatically counterfactual prediction that debt and surplus should be perfectly correlated. With a lot of data and memories of the unit root debates he might fit the correct univariate process, but he would still come to a dramatically counterfactual prediction for debt. As in the analysis of Hansen, Roberds and Sargent (1991), the only way to correctly fit the debt-surplus process in such a way that the value of debt equals the present value of surpluses is to estimate the *joint* debt-surplus process. (And even

[2] One can derive this formula by expressing the surplus as a sum of two AR(1) components, driven by the shocks η.

$$\begin{aligned} s_t &= c_t^* + z_t^* \\ (1-\rho L)c_t^* &= \frac{\beta-\rho}{1-\rho\beta}\eta_t \\ (1-L)z_t^* &= \frac{1-\beta}{1-\beta\rho}\eta_t \end{aligned} \tag{30}$$

One can check that (30) gives the same univariate representation for s_t as (29). Then,

$$v_t = \left(\frac{1}{1-\beta\rho}c_t^* + \frac{1}{1-\beta}z_t^*\right)$$

and one obtains (31) by substituting back for s_t from c_t^* and z_t^*.

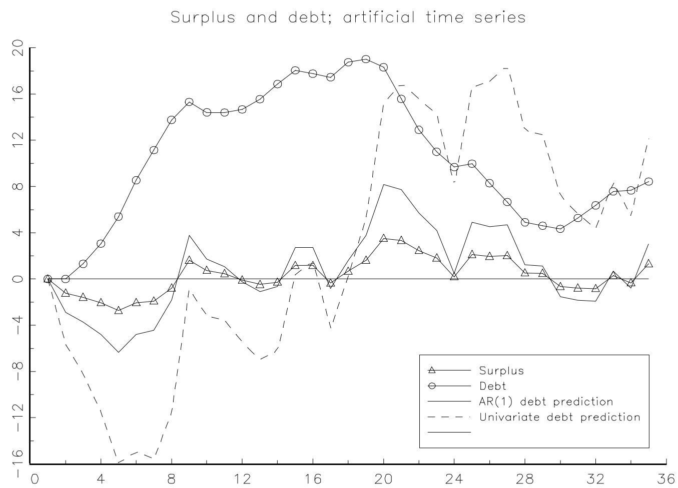

Figure 10: Artificial time series of surplus, real debt, and real debt predicted from the present values of an AR(1) surplus process and the univariate (Wold) surplus process.

this procedure does not test the fiscal theory, since the present value identity holds in both "Ricardian" and "non-Ricardian" regimes, but that's a separate point.)

8 General solution

An exact price level solution for arbitrary debt policy is possible. To find such a general solution, I start with either the flow (3) or present value (4) identities and recursively substitute the same equations for future values of prices p_{t+j}. After some ugly algebra that is relegated to the appendix, the result is

$$p_t = \frac{B_{t-1}(t)}{E_t\left[\sum_{j=0}^{\infty} \beta^j W_{t,j} \quad s_{t+j}\right]}. \tag{32}$$

To define the W terms, first denote the fraction of maturity j debt issued at time t by

$$A_t(t+j) \equiv \frac{B_t(t+j) - B_{t-1}(t+j)}{B_{t+j-1}(t+j)};\ j = 1, 2, ... \tag{33}$$

Then, the W are defined recursively by

$$\begin{aligned} W_{t,0} &= 1 \\ W_{t,1} &= A_t(t+1) \end{aligned} \tag{34}$$

$$\begin{aligned} W_{t,2} &= A_{t+1}(t+2)W_{t,1} + A_t(t+2) \\ W_{t,3} &= A_{t+2}(t+3)W_{t,2} + A_{t+1}(t+3)W_{t,1} + A_t(t+3) \\ W_{t,j} &= \sum_{k=0}^{j-1} A_{t+k}(t+j)W_{t,k}. \end{aligned}$$

To get some sense of what this means, write out the first two terms of the general solution,

$$\begin{aligned} \frac{B_{t-1}(t)}{p_t} &= E_t\left[s_t + \beta\left(1 - \frac{B_{t-1}(t+1)}{B_t(t+1)}\right)s_{t+1} + \right. \\ &\left. +\beta^2\left\{1 - \left[\frac{B_{t-1}(t+1)}{B_t(t+1)}\left(1 - \frac{B_t(t+2)}{B_{t+1}(t+2)}\right) + \frac{B_{t-1}(t+2)}{B_{t+1}(t+2)}\right]\right\}s_{t+2} + ...\right] \end{aligned} \tag{35}$$

The weights $W_{t,j}$ capture the effects of debt policy—the current *and future* maturity structure of the debt—on the relation between the price level and the sequence of surpluses. This general solution delivers the same answer as the special cases which I solved directly above. However, their algebraic complexity motivates the approximate solutions for many applications.

8.1 Linearization about a non-geometric steady state

Above, I linearized the present value identity around a geometric steady state maturity structure to derive an approximate solution. One can also linearize about an arbitrary steady state maturity structure. The terms of this linearization are algebraically complex, as in the general solution. However, these terms need only be evaluated once in defining the steady state, and the approximate solution is then a convenient linear function of surplus and debt policy.

Denote the steady state maturity structure by

$$B_{t-1}(t+j) = B\phi_j.$$

Evaluating (4) at the steady state yields

$$\frac{ps}{B} = (1-\beta)\sum_{j=0}^{\infty}\beta^j\phi_j \equiv \xi \tag{36}$$

which defines the symbol ξ. Denote by $\tilde{B}$ the deviation of debt from the steady state B,

$$\tilde{B}_{t-1}(t+j) = \frac{B_{t-1}(t+j) - \phi_j B}{B}.$$

Now we can differentiate the present value identity (4) to give

$$\sum_{j=0}^{\infty} \beta^j \phi_j E_t \tilde{p}_{t+j} = -\xi \sum_{j=0}^{\infty} \beta^j E_t \tilde{s}_{t+j} + \sum_{j=0}^{\infty} \beta^j \tilde{B}_{t-1}(t+j). \tag{37}$$

The difference between (37) and its counterpart (13) with a geometric steady state is that the terms in expected future prices no longer have a geometric pattern representable by the operator $(1-\beta\phi L^{-1})^{-1}$. Therefore, iterating (13) forward is not pleasant. I substitute the same equation at $t+1, t+2$ etc., and then condense the resulting mass of algebra. The result, presented in the appendix, is

$$\tilde{p}_t \approx -\xi \sum_{j=0}^{\infty} \beta^j W_j \; E_t \tilde{s}_{t+j} - \sum_{j=0}^{\infty} \beta^j D_{j-1} \; B_{t-1+j} \tag{38}$$

where, as above,

$$B_{t-1} \equiv \sum_{j=0}^{\infty} \beta^j \tilde{B}_{t-1}(t+j).$$

The W_j can be interpreted as the steady state level of the general-solution weights $W_{t,j}$,

$$\begin{aligned} W_0 &= 1 \\ W_1 &= A_1 \\ W_2 &= A_1 W_1 + A_2 \\ W_j &= \sum_{k=0}^{j-1} A_{j-k} W_k; \end{aligned}$$

and the A_j can be interpreted as the steady state level of the terms $A_t(t+j)$ in the general solution,

$$A_j \equiv \phi_{j-1} - \phi_j; \tag{39}$$

and the D coefficients are generated from ϕ as the W are generated from the A,

$$\begin{aligned} D_{-1} &= -1 \\ D_0 &= \phi_1 \\ D_1 &= A_1 D_0 + A_2 D_{-1} \\ D_2 &= A_1 D_1 + A_2 D_0 + A_3 D_{-1} \\ D_k &= \sum_{i=1}^{k+1} A_i D_{k-i}. \end{aligned} \tag{40}$$

Despite the appealing recursive structure of the W and D terms, I am not able to find attractive closed form (non-recursive) definitions. However, they only need to be evaluated once, in defining the steady state, where in the general solution they need to be evaluated at each date and state (before taking expectations).

The D coefficients give the effect on the price level at time t of a bond sale at $t+j$ that is then repurchased at $t+j+1$. (Repurchased, because otherwise $B_{t+j+1}(t+k)$ would increase as well as $B_{t+j}(t+k)$.) In principle therefore, D should have a k subscript as well, but it turns out that the answer is the same for all k.

Note that $D_{-2}, D_{-3}... = 0$. Thus, despite the fact that period k debt is sold (and then repurchased), there is no effect on prices past period 1 when the debt is repurchased. Once again, the whole debt policy matters for the price level, including expected future repurchases.

The comparative statics derived above for the geometric case go through. $D_{-1} = -1 < 0$, so selling a little more debt today (period zero) and then buying it back tomorrow (period 1) raises the price level tomorrow. Since $D_0 = \phi_1 \geq 0$, selling a little more debt today can lower prices today (time 0), but only if there is some long term debt outstanding— if $\phi_1 \neq 0$. Interestingly, whether selling a little extra k period debt affects prices today depends on the presence of outstanding time 1 debt, not time k debt.

In general, the terms $D_1,\ D_2...$ are present, so prices at t can be affected by all future expected debt changes. These terms all specialize to zero with the geometric steady state, in which case the price level at t is only affected by B_{t-1} and B_t. To see the effect, then, we need an example in which the maturity structure is far from geometric. Suppose that the steady state maturity structure is $\phi_1 = 1,\ \phi_2 = \phi_3 = ... = 0.5$. The government combines some short term debt or money with some extremely long term debt, for example a perpetuity. Figure 11 plots the response of prices to an anticipated debt sale at time 0, which is then repurchased at time 1, for this case. All the interesting dynamics before time 0 would be absent with a geometric steady state.

9 Conclusion

I started by analyzing the comparative statics of the fiscal theory – the effect of changing surpluses with the debt held constant, and the effect of changing debt with the surplus held constant – while allowing for long-term debt. These comparative statics are quite different from the standard case with only short-term debt. Depending on the maturity structure and debt policy – expectations of future debt sales and repurchases – today's price level can be determined by the present value of all future surpluses, by today's surplus alone, or by a rich variety of intermediate cases. If and only if long-term debt is outstanding, a debt sale can depress the price level today by

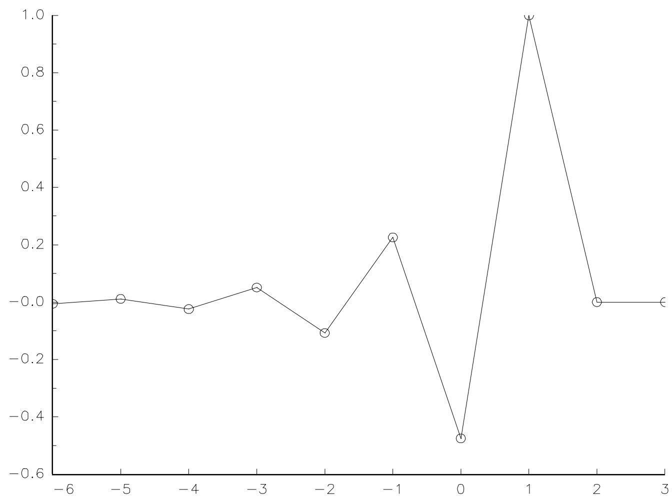

Figure 11: Price path in response to an anticipated debt sale at time 0, which is then repurchased at time 1. The steady state maturity structure is $\phi_1 = 1$, $\phi_2 = \phi_3 = ... = 0.5$,and $\beta = 0.95$.

devaluing outstanding debt.

Then, I considered the question of *optimal* debt and surplus policy in pursuit of stabile inflation. I found that long-term debt can be useful when the present value of surpluses varies by more than surpluses themselves. Perhaps more importantly, long term debt allows the government to offset surplus shocks as they come. In this case, and especially when the government can choose the long-term surplus as well, the optimal policy produces artificial time-series that display many initially puzzling properties of actual time series.

The optimal policies that I study here do not perfectly describe U.S. time series. Their primary failing is that they are too successful: they produce much less variation of inflation than we observe. The optimal active debt plus surplus policy reduces the variance of inflation to zero. The optimal active debt policy leaves some inflation variation, but it is much smaller (relative to variation in the surplus) than we observe in U.S. data, and it has the wrong correlation with the surplus.

One can follow two paths to resolving this issue, both with long histories in the optimal monetary policy literature. Either the problem is harder than the model specifies, or inflation was simply a mistake.

The first path suggests that we add further complications to the models, so that optimal policy produces greater variation in inflation. Most obviously, one could add price stickiness or some other friction. Such frictions would revive the inflation-output trade-offs that were a central part of classical monetary policy analyses such as Sargent and Wallace (1975), and they would generate a serious welfare maximization problem in the modern general equilibrium tradition. Woodford (1996) has analyzed fiscal models with such frictions, and the optimal policy exercises are waiting to be solved.

Alternatively, perhaps inflation was simply a mistake and we should advocate better policy. In a fiscal theory context however, the required policies are not as simple as k-percent rules. Current policy already does a great deal of price stabilization in the face of surplus shocks. For example a k-percent debt growth rule and an AR(1) surplus process leads to prices that have the same proportional variance as does the surplus. The surplus/consumption ratio has, by Figure 2, a standard deviation of about 2 percentage points around an average value of less than one percent. Hence, a k-percent debt rule would lead to inflation with 100% or more standard deviation! For this reason, changing to an explicit rather than implicit commodity standard may be a more practical institutional route to implementing an optimal debt and fiscal policy than trying to more closely implement complex state-contingent debt and surplus choices.

Some interesting behavior is certainly missed by the approximate solutions I use here. For example, in the approximate solutions only the sums $B_{t-1} = \sum_{k=0}^{\infty} \beta^k \tilde{B}_{t-1}(t+k)$ matter to the price level. In a general solution, deliberate state-contingent lengthening and shortening of the maturity structure can affect the time-series process of inflation. However, one must analyze the much more complex general solutions in order to address this interesting question.

Finally, I introduced a very simplistic statistical model in which the "long-run" surplus follows a random walk. The natural direction for an extension of a fiscal theory of the price level is to include the theory of optimal taxation. The properties of the long-term and short-term surplus should be analyzed in a real economic model with distortionary taxation.

10 References

Dupor, Bill, 1997, "Exchange Rates and Bank Notes: The Fiscal Theory of the Price Level" Manuscript, University of Pennsylvania.

Canzoneri, Matthew B., Robert E. Cumby, Behzad T. Diba, 1997, "Is the price level determined by the needs of fiscal solvency?" Working paper, Georgetown University.

Cochrane, John H., 1998, "A Frictionless View of U.S. Inflation," Forthcoming

NBER Macroeconomics Annual, and NBER Working paper 6646.

Hansen, Lars Peter, William Roberds and Thomas J. Sargent, 1991, "Time Series Implications of Present-Value Budget Balance and of Martingale Models of Consumption and Taxes," *Rational Expectations Econometrics*, Westview Press, 121-161.

Leeper, Eric, 1995, "Equilibria Under 'Active' and 'Passive' Monetary Policies," *Journal of Monetary Economics* 27, 129-147.

Lucas, Robert E. Jr., 1972, "Expectations and the Neutrality of Money," *Journal of Economic Theory* 4, 103-124.

Lucas, Robert E. Jr., 1973, "Some International Evidence on Output-Inflation Trade-Offs," *American Economic Review* 63, 326-334.

Sargent, Thomas J. and Neil Wallace, 1975, "'Rational' Expectations, the Optimal Monetary Instrument, and the Optimal Money Supply Rule," *Journal of Political Economy*, 83, 241-254.

Sargent, Thomas J., and Neil Wallace, 1981, "Some Unpleasant Monetarist Arithmetic" *Federal Reserve Bank of Minneapolis Quarterly Review* Fall 1981 (Reprinted in *Rational Expectations and Inflation* New York: HarperCollins College Publishers 1993).

Sims, Christopher A., 1994, "A Simple Model for the Determination of the Price Level and the Interaction of Monetary and Fiscal Policy," *Economic Theory* 4, 381-399.

Sims, Christopher A., 1997, "Fiscal Foundations of Price Stability in Open Economies," Working Paper, Yale University.

Woodford, Michael, 1995, "Price Level Determinacy Without Control of a Monetary Aggregate," *Carnegie-Rochester Conference Series on Public Policy* 43, 1-46.

Woodford, Michael, 1996, "Control of the Public Debt: A Requirement for Price Stability?" *NBER Working Paper 5684.*

Woodford, Michael, 1997, "Doing Without Money: Controlling Inflation in a Post-Monetary World" *NBER Working Paper 6188.*

11 Appendix

11.1 Summary of important notation

s_t = primary (net of interest) surplus.

$p_t =$ price level.

$B_t(j) =$ debt due at j outstanding at the end of period t.

$Q_t(j)$ = nominal price of \$1 face value due at j, as of time t.

r = constant expected gross real return on government debt.

$\beta = 1/r$ = real discount factor.

v_t = real value of the debt.

$p, s, =$ steady state values of s_t, p_t.

B,ϕ steady state debt parameters, $B_{t-1}(t+j) = \phi^j B$.

$\tilde{p}_t = p_t/p - 1$; proportional deviation from steady state.

$\tilde{s}_t = s_t/s - 1$; proportional deviation from steady state.

$\tilde{B}_{t-1}(t+j) = B_{t-1}(t+j)/B - \phi^j$; proportional deviation from steady state.

$B_{t-1} = \sum_{j=0}^{\infty} \beta^j \tilde{B}_{t-1}(t+j)$.

$v_t \equiv \sum_{j=0}^{\infty} \beta^j E_t s_{t+j}$ real value of the debt.

11.2 Flow and present value identities, (3)-(4).

To derive (3)-(4), start with the accounting identity that the primary surplus equals purchases less sales of bonds,

$$B_{t-1}(t) - \sum_{j=1}^{\infty} Q_t(t+j) \left[B_t(t+j) - B_{t-1}(t+j) \right] = p_t s_t. \tag{41}$$

Substituting constant expected real interest rates (2) in (41), we obtain (3). To derive (4) define the real value of the outstanding debt as

$$v_t \equiv \sum_{j=0}^{\infty} \beta^j E_t \left(\frac{1}{p_{t+j}} \right) B_{t-1}(t+j).$$

Then, (3) can be rewritten

$$v_t - \beta E_t v_{t+1} = s_t. \tag{42}$$

Iterating forward, or applying $E_t(1-\beta L^{-1})^{-1}$ to both sides[3], we obtain (4) and vice versa.

[3] This operation also requires the transversality condition that surpluses grow more slowly than the expected bond return. This condition follows if one views the infinite period economies as limits of finite economies, since surpluses are always zero past the last period of a finite economy. Woodford (1995) gives an extensive treatment of weaker transversality conditions in this kind of model.

11.3 Approximation about a geometric steady state, (14)

Define

$$\tilde{v}_t \equiv \sum_{j=0}^{\infty} \beta^j E_t \tilde{s}_{t+j}.$$

and write (13) in lag operator notation,

$$E_t \left(1 - \beta\phi L^{-1}\right)^{-1} \tilde{p}_t = -\left(\frac{1-\beta}{1-\beta\phi}\right) \tilde{v}_t + B_{t-1}$$

Now it is a simple matter to solve for $\tilde{p}_t$ by applying $(1 - \beta\phi L^{-1})$ to both sides. This gives a suggestive and compact version of the price solution,

$$\tilde{p}_t = -\left(\frac{1-\beta}{1-\beta\phi}\right)(\tilde{v}_t - \beta\phi E_t \tilde{v}_{t+1}) + B_{t-1} - \beta\phi B_t. \tag{43}$$

Substituting the definition of $\tilde{v}$ and rearranging gives (14).

11.4 Optimal passive debt policy

Model the surplus as a linear function of a state vector x_t, which may contain lagged values of s_t, by

$$\tilde{s}_t = e' x_t. \tag{44}$$

x_t evolves following a vector $AR(1)$,

$$x_t = A x_{t-1} + J\varepsilon_t; \quad E(\varepsilon\varepsilon') = I \tag{45}$$

Plugging into the price solution (14), the price level is then

$$\tilde{p}_t = -\frac{1-\beta}{1-\beta\phi} e' \left(I - \beta\phi A\right) \left(I - \beta A\right)^{-1} x_t.$$

and the variance of inflation is given by

$$\sigma^2(p_t - p_{t-1}) = \left(\frac{1-\beta}{1-\beta\phi}\right)^2 e' \left(I - \beta\phi A\right) V \left(I - \beta\phi A\right)' e \tag{46}$$

where

$$V = (I - \beta A)^{-1} \Sigma_{\Delta x} (I - \beta A)^{-1\prime}$$

$$\Sigma_{\Delta x} = E\left[(x_t - x_{t-1})(x_t - x_{t-1})'\right] = (A - I) \sum_{j=0}^{\infty} A^j J J' A^{j\prime} (A - I)' + JJ'.$$

Setting to zero the derivative of (46) with respect to ϕ, we find the optimal ϕ from

$$\phi\beta = \frac{e' \left(I - A\right) V e}{e' \left(I - A\right) V A' e}.$$

11.5 Optimal active debt policy, (18) .

I found the solution (18) numerically. I guessed a debt policy of the form

$$(1-\lambda_1 L)(1-\lambda_2 L)B_t = \sum_{j=0}^{N} a_j \tilde{s}_{t-j} = a(L)\tilde{s}_t.$$

For given values of $\phi, \lambda_1, \lambda_2, \{a_j\}$ I calculated the variance of inflation by numerical integration in the frequency domain,

$$\begin{aligned} var\left[(1-L)\tilde{p}_t\right] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1-2\rho\cos(\omega)+\rho^2} f(\omega)f(\omega)^* d\omega \qquad (47) \\ f(\omega) &\equiv (1-e^{-i\omega})k(\phi) - \frac{(1-e^{-i\omega})}{(1-\lambda_1 e^{-i\omega})}\frac{(e^{-i\omega}-\beta\phi)}{(1-\lambda_2 e^{-i\omega})}a(e^{-i\omega}). \end{aligned}$$

I minimized this expression numerically to find the optimal policy $\phi, \lambda_1, \lambda_2, \{a_j\}$. I found the formulas for the coefficients in (18) by staring at the numerical values of the optimal parameters $\lambda_1, \lambda_2, \phi, a_j$ for a large grid of input parameters values ρ, β.

One is tempted to solve (17) and set the variance of inflation to zero by the choice

$$B_t = \frac{1}{\phi\beta}B_{t-1} - \frac{(1-\beta)(1-\beta\phi\rho)}{\beta\phi(1-\beta\rho)(1-\beta\phi)}\tilde{s}_t.$$

The trouble with this idea is that $\beta\phi < 1$ so the coefficient on lagged debt is greater than one. If we solve the difference equation backward, debt is an explosive process. If we solve it forward, debt today must depend on future surpluses, i.e. debt today is not in the information set today. Therefore, we must impose the constraint that B is a non-explosive process, in the time-t information set, and we must expect that constraint to bind. It does; the formulas for variance continue to decline if one allows $\lambda > 1$, so the solution has one root at the constrained value $\lambda = 1$. We proved above that it is not possible to use debt policy alone to achieve a constant price level with fluctuating surpluses, and this discussion is another instance of that fact.

11.6 General price solution, (32)

To simplify notation, let $t = 0$. Define

$$v_t = \sum_{j=0}^{\infty} \beta^j s_{t+j}$$

and define a sequence $\{X_j\}$ by

$$X_0 = 1;$$

$$\begin{aligned}
X_1 &= -\frac{B_{-1}(1)}{B_0(1)} \\
X_2 &= -\frac{B_{-1}(2) + X_1 B_0(2)}{B_1(2)} \\
X_3 &= -\frac{B_{-1}(3) + X_1 B_0(3) + X_2 B_1(3)}{B_2(3)} \\
X_j &= -\sum_{k=0}^{j-1} \frac{B_{k-1}(j)}{B_{j-1}(j)} X_k.
\end{aligned}$$

I start with the present value identity (4), which implies

$$\frac{B_{-1}(0)}{p_0} = E_0 \left\{ v_0 - \beta \left(\frac{1}{p_1} \right) B_{-1}(1) - \beta^2 \left(\frac{1}{p_2} \right) B_{-1}(2) - ... \right\}$$

at time 0 and

$$\frac{1}{p_1} = \frac{1}{B_0(1)} E_1 \left\{ v_1 - \beta \left(\frac{1}{p_2} \right) B_0(2) - \beta^2 \left(\frac{1}{p_3} \right) B_0(3) - ... \right\}$$

at time 1. Use time 1 to substitute in time 0,

$$\frac{B_{-1}(0)}{p_0} = E_0 \left\{ v_0 - \beta \frac{B_{-1}(1)}{B_0(1)} \left[v_1 - \beta \frac{1}{p_2} B_0(2) - ... \right] - \beta^2 \left(\frac{1}{p_2} \right) B_{-1}(2) - ... \right\}$$

Recognizing the definition of X_1

$$\begin{aligned}
\frac{B_{-1}(0)}{p_0} &= E_0 \left\{ v_0 + \beta X_1 v_1 + \beta^2 \left[-\left(B_{-1}(2) + X_1 B_0(2) \right) \frac{1}{p_2} \right] + \right. \\
&\quad \left. + \beta^3 \left[-\left(B_{-1}(3) + X_1 B_0(3) \right) \frac{1}{p_3} \right] + ... \right\}.
\end{aligned}$$

Substitute now for $1/p_2$.

$$\frac{1}{p_2} = \frac{1}{B_1(2)} E_2 \left[v_2 - \beta \frac{1}{p_3} B_1(3) - \beta^2 \frac{1}{p_4} B_1(4) ... \right]$$

$$\begin{aligned}
\frac{B_{-1}(0)}{p_0} &= E_0 \left\{ v_0 + \beta X_1 v_1 + \beta^2 \left[-\frac{B_{-1}(2) + X_1 B_0(2)}{B_1(2)} \left[v_2 - \beta \frac{1}{p_3} B_1(3) - ... \right] \right] + \right. \\
&\quad \left. + \beta^3 \left[-\left(B_{-1}(3) + X_1 B_0(3) \right) \frac{1}{p_3} \right] + ... \right\}.
\end{aligned}$$

Recognizing the definition of X_2

$$\frac{B_{-1}(0)}{p_0} = E_0 \left\{ v_0 + \beta X_1 v_1 + \beta^2 X_2 v_2 - \beta^3 \left(\left[B_{-1}(3) + X_1 B_0(3) + X_2 B_1(3) \right] \frac{1}{p_3} \right) + ... \right\}.$$

Continuing in this way, we have

$$\frac{B_{-1}(0)}{p_0} = E_0 \sum_{j=0}^{\infty} \beta^j X_j v_j.$$

This expression is already a solution. However, it is more elegant to collect terms in s_j on the right hand side, resulting in

$$\frac{B_{-1}(0)}{p_0} = E_0 \left\{ 1 + (1 + X_1)\,\beta s_1 + (1 + X_1 + X_2)\,\beta^2 s_2 + ... \right\}$$

$$\frac{B_{-1}(0)}{p_0} = E_0 \sum_{j=0}^{\infty} \beta^j \left(\sum_{k=0}^{j} X_k \right) s_j = \sum_{j=0}^{\infty} \beta^j W_j s_j.$$

This is the price solution (32). The last equality defines W_j. We can find a more direct definition for W_j rather than via X_j. Proceeding through time,

$$W_0 = X_0 = 1$$

$$W_1 = 1 + X_1 = \frac{B_0(1) - B_{-1}(1)}{B_0(1)} = A_0(1)$$

$$\begin{aligned} W_2 &= 1 + X_1 + X_2 = W_1 + X_2 = \frac{W_1 B_1(2) - X_1 B_0(2) - B_{-1}(2)}{B_1(2)} = \\ &= \frac{W_1 B_1(2) - (W_1 - 1) B_0(2) - B_{-1}(2)}{B_1(2)} = W_1 A_1(2) + A_0(2). \end{aligned}$$

$$\begin{aligned} W_3 &= W_2 + X_3 = \frac{W_2 B_3(3) - X_2 B_2(3) - X_1 B_1(3) - B_0(3)}{B_3(3)} = \\ &= \frac{W_2 B_2(3) + (W_1 - W_2) B_1(3) + (1 - W_1) B_0(3) - B_{-1}(3)}{B_2(3)} = \\ &= W_2 A_2(3) + W_1 A_1(3) + A_0(3) \end{aligned}$$

and so forth.

11.7 Approximate solution with a non-geometric steady state (38)

We can differentiate the general solution (32) with respect to debt at the steady state to find that the approximation for the s terms is

$$\tilde{p}_t \approx -\xi \sum_{j=0}^{\infty} \beta^j W_j \; E_t \tilde{s}_{t+j}.$$

where, again $\xi = ps/B$. The hard part is to unravel the $W_{t,j}$ terms to find the effects of $dB_{t+j}(t+k)$. We could proceed directly by differentiating $W_{t,j}$, or perform the same sort of nonlinear forward iteration on the approximated present value identity. It turns out to be easiest to track the effects on p_t of a single debt operation $dB_{t+j}(t+k)$.

We want to evaluate the effect of a small change in $B_0(j)$. This is a sale of a little extra date j debt at time 0, followed by a repurchase of that debt at time 1. We start with the real time t flow identity, (3), which I repeat here for convenience.

$$s_t + \sum_{j=1} \beta^j E_t \left(\frac{1}{p_{t+j}}\right) [B_t(t+j) - B_{t-1}(t+j)] = \frac{B_{t-1}(t)}{p_t}.$$

$B_0(j)$ does not enter this identity, or the general solution, for $t \geq 2$, so prices at and past date 2 are not affected. Prices are forward-looking and so are affected only by current and future debt policy.

At time 1, the flow identity is

$$s_1 + \beta E_1 \left(\frac{1}{p_2}\right) [B_1(2) - B_0(2)] + .. + \beta^{j-1} E_1 \left(\frac{1}{p_j}\right) [B_1(j) - B_0(j)] + ... = \frac{B_0(1)}{p_1}.$$

We take the derivative of this identity with respect to $B_0(j)$, evaluated at the steady state. The result is

$$D_1 \equiv \frac{pB}{\beta^{j-1}} \frac{d\,(1/p_1)}{dB_0(j)} = -1.$$

The time 0 constraint is

$$s_0 + \beta E_0 \left(\frac{1}{p_1}\right) [B_0(1) - B_{-1}(1)] + .. + \beta^j E_0 \left(\frac{1}{p_j}\right) [B_0(j) - B_{-1}(j)] + ... = \frac{B_{-1}(0)}{p_0}.$$

Taking the derivative with respect to $B_0(j)$ again,

$$\beta B\,(1-\phi_1)\, \frac{d\,(1/p_1)}{dB_0(j)} + \beta^j \frac{1}{p} = B \frac{d\,(1/p_0)}{dB_0(j)}$$

$$D_0 = (1-\phi_1)\, D_1 + 1 = \phi.$$

At a generic time $-k$, the flow identity is

$$s_{-k} + \sum_{i=1} \beta^i E_{-k} \left(\frac{1}{p_{-k+i}}\right) [B_{-k}(-k+i) - B_{-k-1}(-k+i)] = \frac{1}{p_{-k}} [-B_{-k-1}(-k)]\,.$$

Taking derivatives with respect to $B_0(j)$ again,

$$\sum_{i=1}^{k+1} \beta^i \frac{d\,(1/p_{-k+i})}{dB_0(j)} [\phi_{i-1} - \phi_i] = \frac{d\,(1/p_{-k})}{dB_0(j)}$$

$$D_{-k} = \sum_{i=1}^{k+1} D_{-k+i} A_i.$$

Plugging these derivatives in, we obtain the linearization (38).